Knapsack Problem by DP

Given n items of

integer weights: w_1 w_2 … w_n

values: v_1 v_2 … v_n

a knapsack of integer capacity W

find most valuable subset of the items that fit into the knapsack

Consider instance defined by first i items and capacity j ($j \leq W$).

Let $V[i,j]$ be optimal value of such instance. Then

$$V[i,j] = \begin{cases} \max \{V[i\text{-}1,j], v_i + V[i\text{-}1,j\text{-} w_i]\} & \text{if } j\text{-} w_i \geq 0 \\ V[i\text{-}1,j] & \text{if } j\text{-} w_i < 0 \end{cases}$$

Initial conditions: $V[0,j] = 0$ and $V[i,0] = 0$

1. Among the subsets that do not include the ith item, the value of an optimal subset is, by definition, $F(i-1, j)$.
2. Among the subsets that do include the ith item (hence, $j - w_i \geq 0$), an optimal subset is made up of this item and an optimal subset of the first $i-1$ items that fits into the knapsack of capacity $j - w_i$. The value of such an optimal subset is $v_i + F(i-1, j-w_i)$.

Recurrence equation

$$F(i, j) = \begin{cases} \max\{F(i-1, j),\ v_i + F(i-1, j-w_i)\} & \text{if } j - w_i \geq 0, \\ F(i-1, j) & \text{if } j - w_i < 0. \end{cases}$$

$$F(0, j) = 0 \text{ for } j \geq 0 \quad \text{and} \quad F(i, 0) = 0 \text{ for } i \geq 0.$$

		0	$j-w_i$	j	W
	0	0	0	0	0
	$i-1$	0	$F(i-1, j-w_i)$	$F(i-1, j)$	
w_i, v_i	i	0		$F(i, j)$	
	n	0			goal

	i	capacity j 0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2,\ v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1,\ v_2 = 10$	2	0	10	12	22	22	22
$w_3 = 3,\ v_3 = 20$	3	0	10	12	22	30	32
$w_4 = 2,\ v_4 = 15$	4	0	10	15	25	30	**37**

Knapsack Problem by DP (example)

Example: Knapsack of capacity $W = 5$

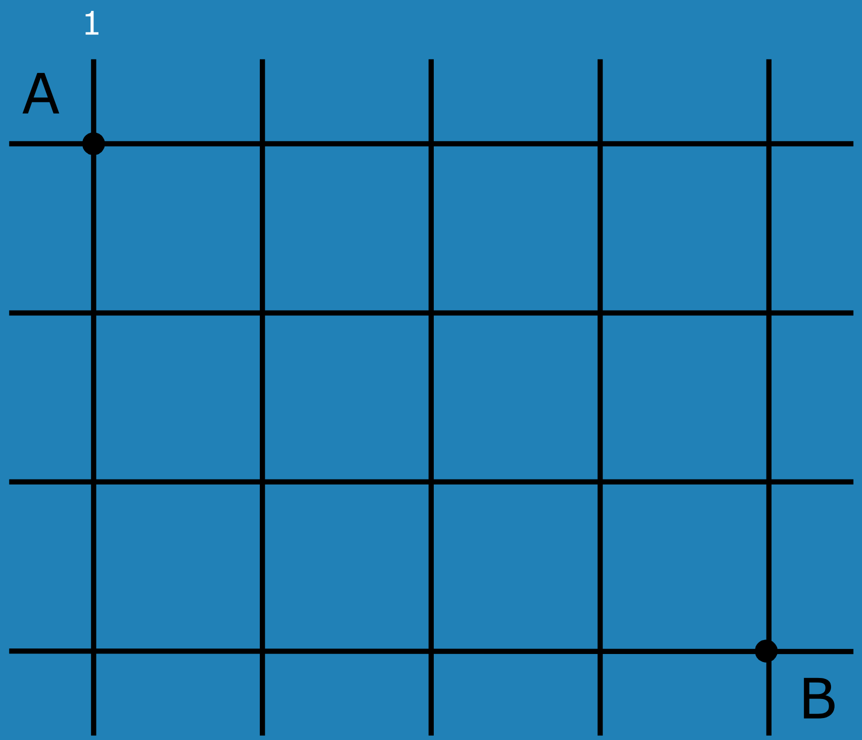
item	weight	value
1	2	$12
2	1	$10
3	3	$20
4	2	$15

capacity j

		0	1	2	3	4	5
	0						
$w_1 = 2, v_1 = 12$	1						
$w_2 = 1, v_2 = 10$	2						
$w_3 = 3, v_3 = 20$	3						
$w_4 = 2, v_4 = 15$	4						?

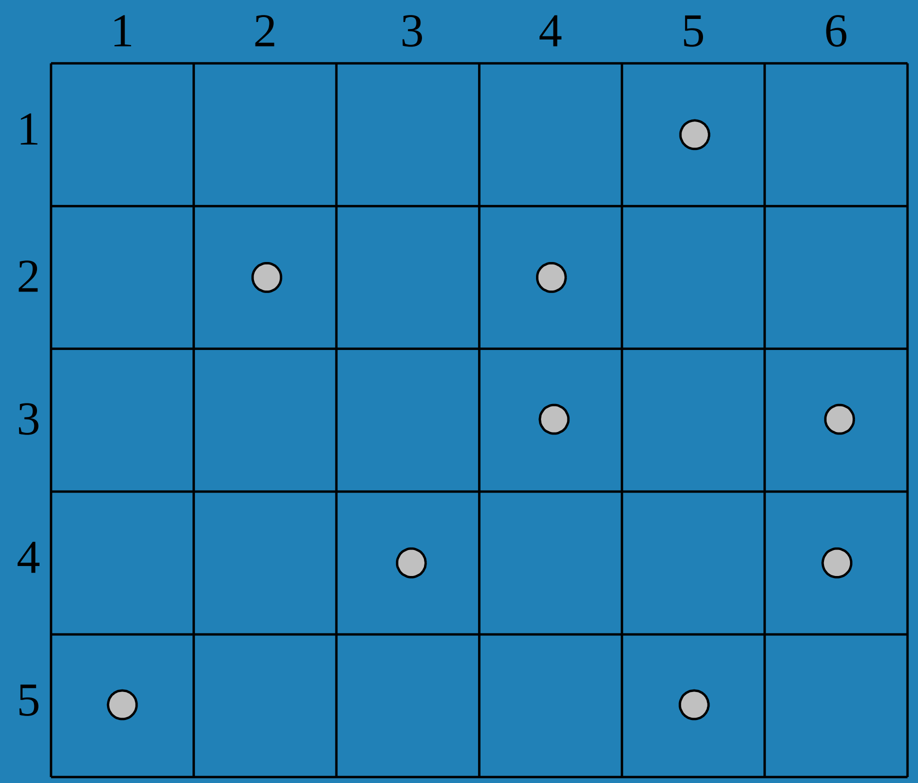
Example 3: Path counting

Consider the problem of counting the number of shortest paths from point A to point B in a city with perfectly horizontal streets and vertical avenues



Example 4: Coin-collecting by robot

Several coins are placed in cells of an $n \times m$ board. A robot, located in the upper left cell of the board, needs to collect as many of the coins as possible and bring them to the bottom right cell. On each step, the robot can move either one cell to the right or one cell down from its current location.



Solution to the coin-collecting problem

Let $F(i,j)$ be the largest number of coins the robot can collect and bring to cell (i,j) in the ith row and jth column.

The largest number of coins that can be brought to cell (i,j):

from the left neighbor ?
from the neighbor above?

The recurrence:

$$F(i, j) = \max\{F(i-1, j),\ F(i, j-1)\} + c_{ij} \quad \text{for } 1 \le i \le n,\ 1 \le j \le m$$

where $c_{ij} = 1$ if there is a coin in cell (i,j), and $c_{ij} = 0$ otherwise

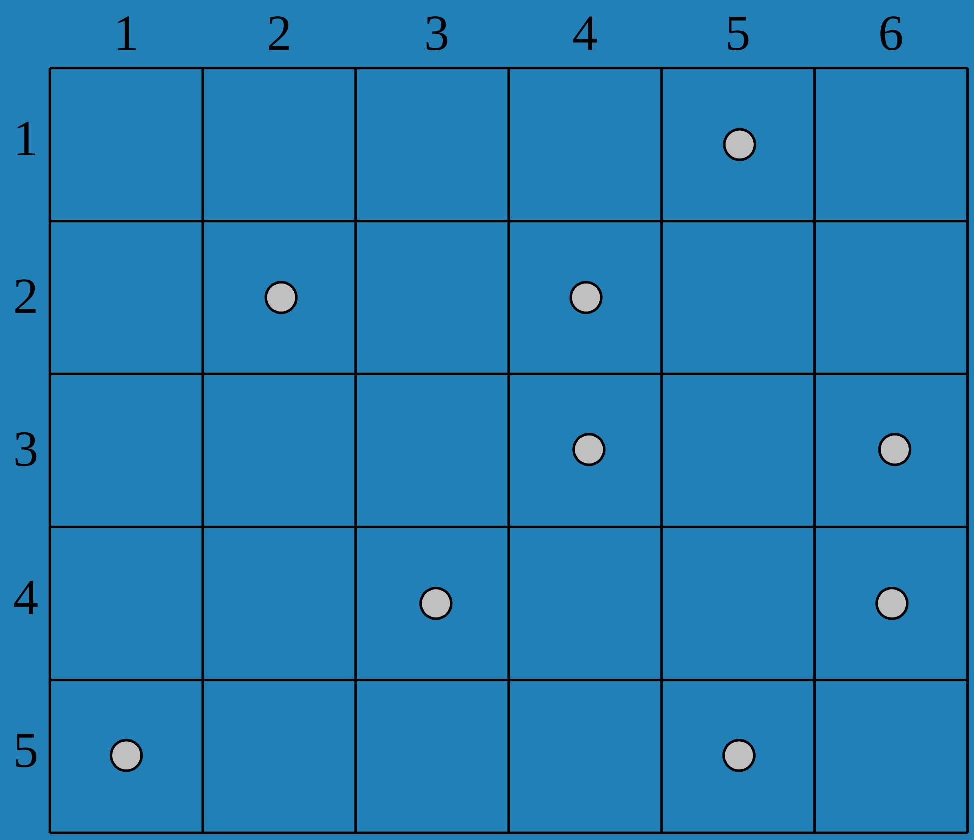
$$F(0, j) = 0 \text{ for } 1 \le j \le m \text{ and } F(i, 0) = 0 \text{ for } 1 \le i \le n.$$

Solution to the coin-collecting problem (cont.)

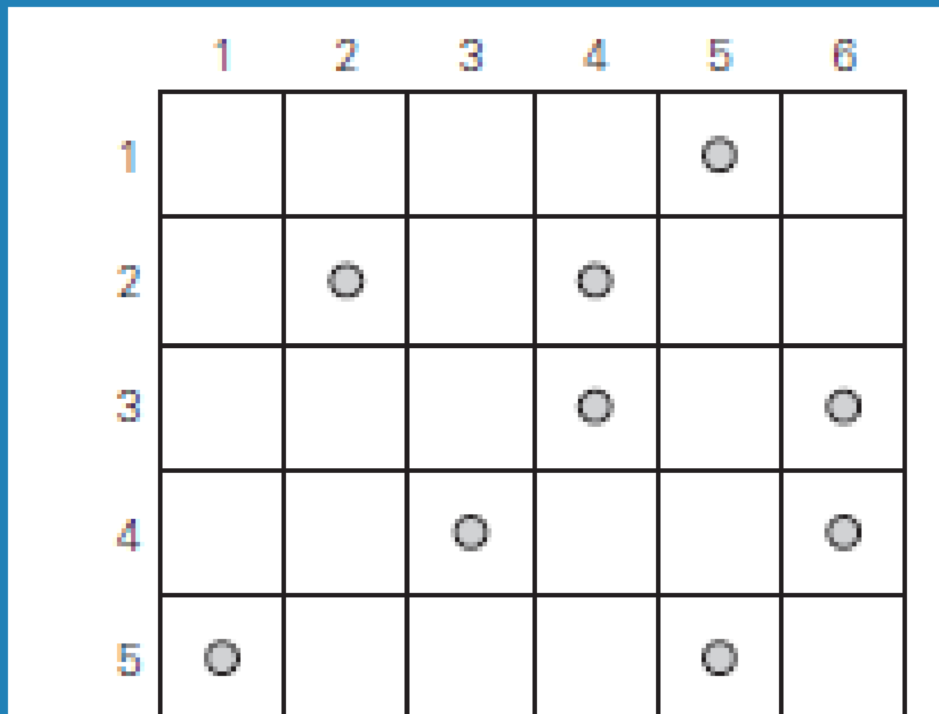
$F(i, j) = \max\{F(i-1, j),\ F(i, j-1)\} + c_{ij}$ for $1 \le i \le n$, $1 \le j \le m$

where $c_{ij} = 1$ if there is a coin in cell (i,j), and $c_{ij} = 0$ otherwise

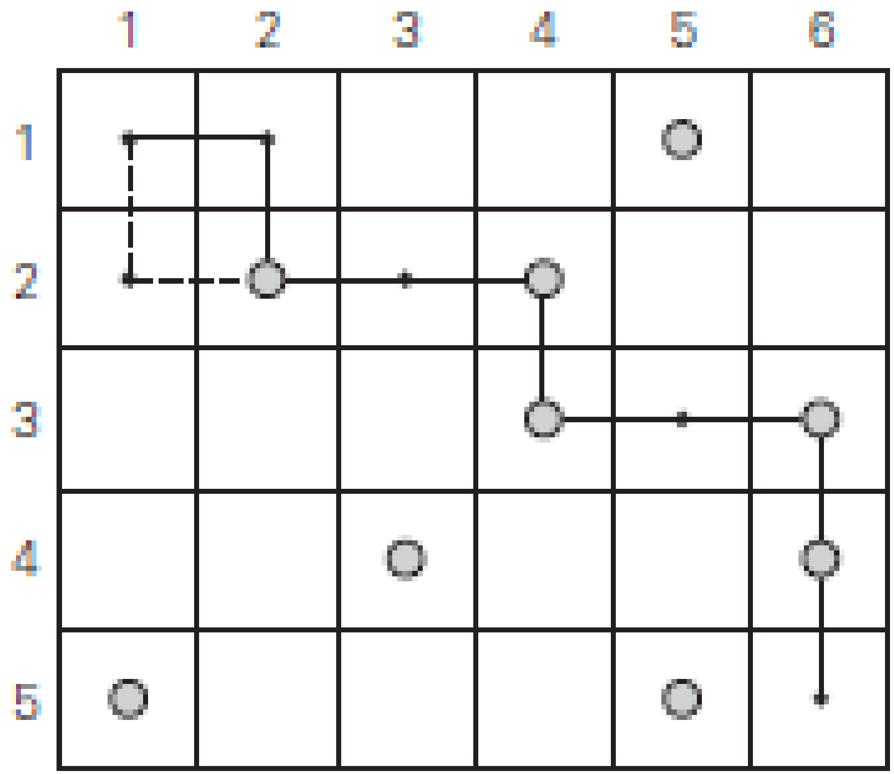
$F(0, j) = 0$ for $1 \le j \le m$ and $F(i, 0) = 0$ for $1 \le i \le n$.



Example



	1	2	3	4	5	6
1	0	0	0	0	1	1
2	0	1	1	2	2	2
3	0	1	1	3	3	4
4	0	1	2	3	3	5
5	1	1	2	3	4	5


1
2
3
4
5
6
1
2
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4
5