

# Stable Marriage Problem

# Stable Marriage Problem

In this section, we consider an interesting version of bipartite matching called the stable marriage problem. Consider a set  $Y = \{m_1, m_2, \dots, m_n\}$  of  $n$  men and a set  $X = \{w_1, w_2, \dots, w_n\}$  of  $n$  women. Each man has a preference list ordering the women as potential marriage partners with no ties allowed. Similarly, each woman has a preference list of the men, also with no ties. Examples of these two

| men's preferences |     |     |     | women's preferences |     |     |     | ranking matrix |   |   |   |
|-------------------|-----|-----|-----|---------------------|-----|-----|-----|----------------|---|---|---|
|                   | 1st | 2nd | 3rd |                     | 1st | 2nd | 3rd |                | Ann   | Lea   | Sue   |
| Bob:              | Lea | Ann | Sue | Ann:                | Jim | Tom | Bob | Bob            | <span style="border: 1px solid black;">2,3</span> | 1,2   | 3,3   |
| Jim:              | Lea | Sue | Ann | Lea:                | Tom | Bob | Jim | Jim            | 3,1   | <span style="border: 1px solid black;">1,3</span> | 2,1   |
| Tom:              | Sue | Lea | Ann | Sue:                | Jim | Tom | Bob | Tom            | 3,2   | 2,1   | <span style="border: 1px solid black;">1,2</span> |
|                   | (a) |     |     |                     | (b) |     |     |                | (c)   |   |   |

- A **marriage matching  $M$**  is a set of  $n$   $(m, w)$  pairs whose members **are selected** from disjoint  $n$ -element sets  $Y$  and  $X$  in a one-one fashion, i.e., each man  $m$  from  $Y$  is paired with exactly one woman  $w$  from  $X$  and vice versa.
- A pair  $(m, w)$ , where  $m \in Y$ ,  $w \in X$ , is said to be a blocking pair for a marriage matching  $M$  if man  $m$  and woman  $w$  are not matched in  $M$  but they prefer each other to their mates in  $M$ .

- A marriage matching  $M$  is called ***stable if there is no blocking*** pair for it; otherwise,  $M$  is called ***unstable***.
- The ***stable marriage problem*** is to find a stable marriage matching for men's and women's given preferences.

## Stable marriage algorithm

Input: A set of  $n$  men and a set of  $n$  women along with rankings of the women by each man and rankings of the men by each woman with no ties allowed in the rankings

Output: A stable marriage matching

**Step 0** Start with all the men and women being free.

**Step 1** While there are free men, arbitrarily select one of them and do the following:

*Proposal* The selected free man  $m$  proposes to  $w$ , the next woman on his preference list (who is the highest-ranked woman who has not rejected him before).

*Response* If  $w$  is free, she accepts the proposal to be matched with  $m$ . If she is not free, she compares  $m$  with her current mate. If she prefers  $m$  to him, she accepts  $m$ 's proposal, making her former mate free; otherwise, she simply rejects  $m$ 's proposal, leaving  $m$  free.

**Step 2** Return the set of  $n$  matched pairs.

|                            |     |   |   |   |  |
|----------------------------|-----|---|---|---|--|
|                            |     | Ann   | Lea   | Sue   |  |
| Free men:<br>Bob, Jim, Tom | Bob | 2, 3  | <span style="border: 1px solid black;">1,2</span> | 3, 3  | Bob proposed to Lea<br>Lea accepted              |
|                            | Jim | 3, 1  | 1, 3  | 2, 1  |  |
|                            | Tom | 3, 2  | 2, 1  | 1, 2  |  |
|                            |     | Ann   | Lea   | Sue   |  |
| Free men:<br>Jim, Tom      | Bob | 2, 3  | <span style="border: 1px solid black;">1,2</span> | 3, 3  | Jim proposed to Lea<br>Lea rejected              |
|                            | Jim | 3, 1  | <u>1, 3</u>                                       | 2, 1  |  |
|                            | Tom | 3, 2  | <u>2, 1</u>                                       | 1, 2  |  |
|                            |     | Ann   | Lea   | Sue   |  |
| Free men:<br>Jim, Tom      | Bob | 2, 3  | <span style="border: 1px solid black;">1,2</span> | 3, 3  | Jim proposed to Sue<br>Sue accepted              |
|                            | Jim | 3, 1  | 1, 3  | <span style="border: 1px solid black;">2,1</span> |  |
|                            | Tom | 3, 2  | 2, 1  | 1, 2  |  |
|                            |     | Ann   | Lea   | Sue   |  |
| Free men:<br>Tom           | Bob | 2, 3  | <span style="border: 1px solid black;">1,2</span> | 3, 3  | Tom proposed to Sue<br>Sue rejected              |
|                            | Jim | 3, 1  | 1, 3  | <span style="border: 1px solid black;">2,1</span> |  |
|                            | Tom | 3, 2  | 2, 1  | <u>1, 2</u>                                       |  |
|                            |     | Ann   | Lea   | Sue   |  |
| Free men:<br>Tom           | Bob | 2, 3  | 1, 2  | 3, 3  | Tom proposed to Lea<br>Lea replaced Bob with Tom |
|                            | Jim | 3, 1  | 1, 3  | <span style="border: 1px solid black;">2,1</span> |  |
|                            | Tom | 3, 2  | <span style="border: 1px solid black;">2,1</span> | 1, 2  |  |
|                            |     | Ann   | Lea   | Sue   |  |
| Free men:<br>Bob           | Bob | <span style="border: 1px solid black;">2,3</span> | 1, 2  | 3, 3  | Bob proposed to Ann<br>Ann accepted              |
|                            | Jim | 3, 1  | 1, 3  | <span style="border: 1px solid black;">2,1</span> |  |
|                            | Tom | 3, 2  | <span style="border: 1px solid black;">2,1</span> | 1, 2  |  |

**FIGURE 10.12** Application of the stable marriage algorithm. An accepted proposal is indicated by a boxed cell; a rejected proposal is shown by an underlined cell.

|       | woman 1 | woman 2 |
|-------|---------|---------|
| man 1 | 1, 2    | 2, 1    |
| man 2 | 2, 1    | 1, 2    |

Men proposing version, women proposing

|          | <i>A</i> | <i>B</i> | <i>C</i> |
|----------|----------|----------|----------|
| $\alpha$ | 1, 3     | 2, 2     | 3, 1     |
| $\beta$  | 3, 1     | 1, 3     | 2, 2     |
| $\gamma$ | 2, 2     | 3, 1     | 1, 3     |



|                         |          |        |        |        |                          |
|-------------------------|----------|--------|--------|--------|--------------------------|
| Free men:               | $\alpha$ | $A$    | $B$    | $C$    |                          |
| $\alpha, \beta, \gamma$ | $\beta$  | $3, 1$ | $1, 3$ | $2, 2$ | $\alpha$ proposed to $A$ |
|                         | $\gamma$ | $2, 2$ | $3, 1$ | $1, 3$ | $A$ accepted             |

|                 |          |        |        |        |                         |
|-----------------|----------|--------|--------|--------|-------------------------|
| Free men:       | $\alpha$ | $A$    | $B$    | $C$    |                         |
| $\beta, \gamma$ | $\beta$  | $3, 1$ | $1, 3$ | $2, 2$ | $\beta$ proposed to $B$ |
|                 | $\gamma$ | $2, 2$ | $3, 1$ | $1, 3$ | $B$ accepted            |

|           |          |        |        |        |                          |
|-----------|----------|--------|--------|--------|--------------------------|
| Free men: | $\alpha$ | $A$    | $B$    | $C$    |                          |
| $\gamma$  | $\beta$  | $3, 1$ | $1, 3$ | $2, 2$ | $\gamma$ proposed to $C$ |
|           | $\gamma$ | $2, 2$ | $3, 1$ | $1, 3$ | $C$ accepted             |

the (man-optimal) stable marriage matching is  $M = \{(\alpha, A), (\beta, B), (\gamma, C)\}$ .

|                |          |          |          |          |                              |
|----------------|----------|----------|----------|----------|------------------------------|
| Free women:    |          | <i>A</i> | <i>B</i> | <i>C</i> |                              |
| <i>A, B, C</i> | $\alpha$ | 1,3      | 2,2      | 3,1      | <i>A</i> proposed to $\beta$ |
|                | $\beta$  | 3,1      | 1,3      | 2,2      | $\beta$ accepted             |
|                | $\gamma$ | 2,2      | 3,1      | 1,3      |                              |

|             |          |          |          |          |                               |
|-------------|----------|----------|----------|----------|-------------------------------|
| Free women: |          | <i>A</i> | <i>B</i> | <i>C</i> |                               |
| <i>B, C</i> | $\alpha$ | 1,3      | 2,2      | 3,1      | <i>B</i> proposed to $\gamma$ |
|             | $\beta$  | 3,1      | 1,3      | 2,2      | $\gamma$ accepted             |
|             | $\gamma$ | 2,2      | 3,1      | 1,3      |                               |

|             |          |          |          |          |                               |
|-------------|----------|----------|----------|----------|-------------------------------|
| Free women: |          | <i>A</i> | <i>B</i> | <i>C</i> |                               |
| <i>C</i>    | $\alpha$ | 1,3      | 2,2      | 3,1      | <i>C</i> proposed to $\alpha$ |
|             | $\beta$  | 3,1      | 1,3      | 2,2      | $\alpha$ accepted             |
|             | $\gamma$ | 2,2      | 3,1      | 1,3      |                               |

(woman-optimal) stable marriage matching is  $M = \{(\beta, A), (\gamma, B), (\alpha, C)\}$ .

|          | <i>A</i> | <i>B</i> | <i>C</i> | <i>D</i> |
|----------|----------|----------|----------|----------|
| $\alpha$ | 1, 3     | 2, 3     | 3, 2     | 4, 3     |
| $\beta$  | 1, 4     | 4, 1     | 3, 4     | 2, 2     |
| $\gamma$ | 2, 2     | 1, 4     | 3, 3     | 4, 1     |
| $\delta$ | 4, 1     | 2, 2     | 3, 1     | 1, 4     |

iteration 1

Free men:  $\alpha, \beta, \gamma, \delta$

|          | <i>A</i>  | <i>B</i> | <i>C</i> | <i>D</i> |
|----------|---|----------|----------|----------|
| $\alpha$ | <span style="border: 1px solid black;">1,3</span> | 2,3      | 3,2      | 4,3      |
| $\beta$  | 1,4   | 4,1      | 3,4      | 2,2      |
| $\gamma$ | 2,2   | 1,4      | 3,3      | 4,1      |
| $\delta$ | 4,1   | 2,2      | 3,1      | 1,4      |

$\alpha$  proposed to *A*; *A* accepted

iteration 2

Free men:  $\beta, \gamma, \delta$

|          | <i>A</i>  | <i>B</i> | <i>C</i> | <i>D</i> |
|----------|---|----------|----------|----------|
| $\alpha$ | <span style="border: 1px solid black;">1,3</span> | 2,3      | 3,2      | 4,3      |
| $\beta$  | <span style="border: 1px solid black;">1,4</span> | 4,1      | 3,4      | 2,2      |
| $\gamma$ | 2,2   | 1,4      | 3,3      | 4,1      |
| $\delta$ | 4,1   | 2,2      | 3,1      | 1,4      |

$\beta$  proposed to *A*; *A* rejected

iteration 3

Free men:  $\beta, \gamma, \delta$

|          | <i>A</i>  | <i>B</i> | <i>C</i> | <i>D</i>  |
|----------|---|----------|----------|---|
| $\alpha$ | <span style="border: 1px solid black;">1,3</span> | 2,3      | 3,2      | 4,3   |
| $\beta$  | 1,4   | 4,1      | 3,4      | <span style="border: 1px solid black;">2,2</span> |
| $\gamma$ | 2,2   | 1,4      | 3,3      | 4,1   |
| $\delta$ | 4,1   | 2,2      | 3,1      | 1,4   |

$\beta$  proposed to *D*; *D* accepted

iteration 4

Free men:  $\gamma, \delta$

|          | <i>A</i>  | <i>B</i>  | <i>C</i> | <i>D</i>  |
|----------|---|---|----------|---|
| $\alpha$ | <span style="border: 1px solid black;">1,3</span> | 2,3   | 3,2      | 4,3   |
| $\beta$  | 1,4   | 4,1   | 3,4      | <span style="border: 1px solid black;">2,2</span> |
| $\gamma$ | 2,2   | <span style="border: 1px solid black;">1,4</span> | 3,3      | 4,1   |
| $\delta$ | 4,1   | 2,2   | 3,1      | 1,4   |

$\gamma$  proposed to *B*; *B* accepted

iteration 5  
Free men:  $\delta$

|          | A   | B   | C   | D   |
|----------|-----|-----|-----|-----|
| $\alpha$ | 1,3 | 2,3 | 3,2 | 4,3 |
| $\beta$  | 1,4 | 4,1 | 3,4 | 2,2 |
| $\gamma$ | 2,2 | 1,4 | 3,3 | 4,1 |
| $\delta$ | 4,1 | 2,2 | 3,1 | 1,4 |

$\delta$  proposed to D; D rejected

iteration 7  
Free men:  $\gamma$

|          | A   | B   | C   | D   |
|----------|-----|-----|-----|-----|
| $\alpha$ | 1,3 | 2,3 | 3,2 | 4,3 |
| $\beta$  | 1,4 | 4,1 | 3,4 | 2,2 |
| $\gamma$ | 2,2 | 1,4 | 3,3 | 4,1 |
| $\delta$ | 4,1 | 2,2 | 3,1 | 1,4 |

proposed to A; A replaced  $\alpha$  with  $\gamma$

iteration 9  
Free men:  $\alpha$

|          | A   | B   | C   | D   |
|----------|-----|-----|-----|-----|
| $\alpha$ | 1,3 | 2,3 | 3,2 | 4,3 |
| $\beta$  | 1,4 | 4,1 | 3,4 | 2,2 |
| $\gamma$ | 2,2 | 1,4 | 3,3 | 4,1 |
| $\delta$ | 4,1 | 2,2 | 3,1 | 1,4 |

$\alpha$  proposed to C; C accepted

iteration 6  
Free men:  $\delta$

|          | A   | B   | C   | D   |
|----------|-----|-----|-----|-----|
| $\alpha$ | 1,3 | 2,3 | 3,2 | 4,3 |
| $\beta$  | 1,4 | 4,1 | 3,4 | 2,2 |
| $\gamma$ | 2,2 | 1,4 | 3,3 | 4,1 |
| $\delta$ | 4,1 | 2,2 | 3,1 | 1,4 |

$\delta$  proposed to B; B replaced  $\gamma$  with  $\delta$

iteration 8  
Free men:  $\alpha$

|          | A   | B   | C   | D   |
|----------|-----|-----|-----|-----|
| $\alpha$ | 1,3 | 2,3 | 3,2 | 4,3 |
| $\beta$  | 1,4 | 4,1 | 3,4 | 2,2 |
| $\gamma$ | 2,2 | 1,4 | 3,3 | 4,1 |
| $\delta$ | 4,1 | 2,2 | 3,1 | 1,4 |

$\alpha$  proposed to B; B rejected

Free men: none

$$M = \{(\alpha, C), (\beta, D), (\gamma, A), (\delta, B)\}$$

**a.** in the worst case.

**b.** in the best case.