Simplex Method

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Iterative Improvement

Algorithm design technique for solving optimization problems

- Start with a feasible solution
- Repeat the following step until no improvement can be found:
  - change the current feasible solution to a feasible solution with a better value of the objective function
- Return the last feasible solution as optimal

Note: Typically, a change in a current solution is "small" (*local search*)

Major difficulty: Local optimum vs. global optimum

Examples

- simplex method
- Ford-Fulkerson algorithm for maximum flow problem
- maximum matching of graph vertices
- Gale-Shapley algorithm for the stable marriage problem
- local search heuristics

Linear Programming

Linear programming (LP) problem is to optimize a linear function of several variables subject to linear constraints:

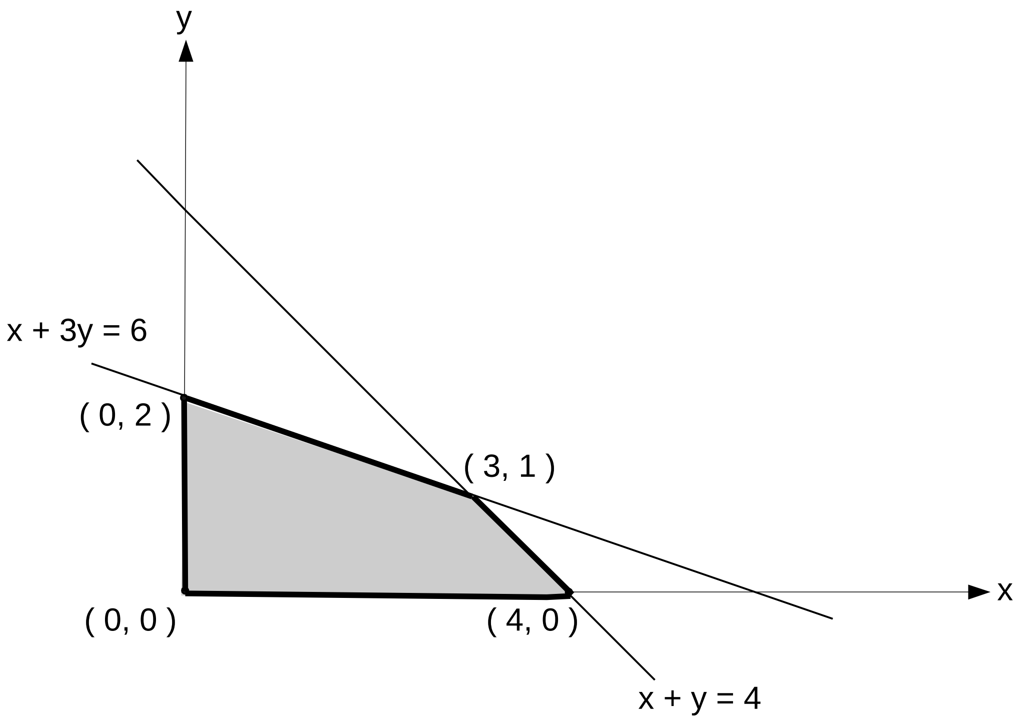
maximize (or minimize) $c_1 x_1 + ... + c_n x_n$
subject to $a_{i1}x_1 + ... + a_{in} x_n \le (\text{or} \ge \text{or} =)\ b_i$,
$i = 1,...,m$ $x_1 \ge 0, ... , x_n \ge 0$

The function $z = c_1 x_1 + ... + c_n x_n$ is called the *objective function*;

constraints $x_1 \ge 0, ... , x_n \ge 0$ are called *nonnegativity constraints*

$$
\begin{aligned}
\text{maximize} \quad & 3x + 5y \\
\text{subject to} \quad & x + y \le 4 \\
& x + 3y \le 6 \\
& x \ge 0, \; y \ge 0
\end{aligned}
$$

Feasible Region



3 possible outcomes in solving an LP problem

- has a finite optimal solution, which may not be unique

- *unbounded*: the objective function of maximization (minimization) LP problem is unbounded from above (below) on its feasible region

- *infeasible*: there are no points satisfying all the constraints, i.e. the constraints are contradictory

$$\begin{aligned} \text{maximize} \quad & 3x + 5y \\ \text{subject to} \quad & x + y >= 4 \\ & x + 3y >= 6 \\ & x \geq 0, \; y \geq 0 \end{aligned}$$

THEOREM 9.1 Optimal Solution of a Linear Programming Problem

If a linear programming problem has an optimal solution, then it must occur at a vertex of the set of feasible solutions. If the problem has more than one optimal solution, then at least one of them must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

Graphical Method for Solving a Linear Programming Problem

To solve a linear programming problem involving two variables by the graphical method, use the steps listed below.

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions*.)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, *if* an optimal solution exists, then it will occur at a vertex.)

Example

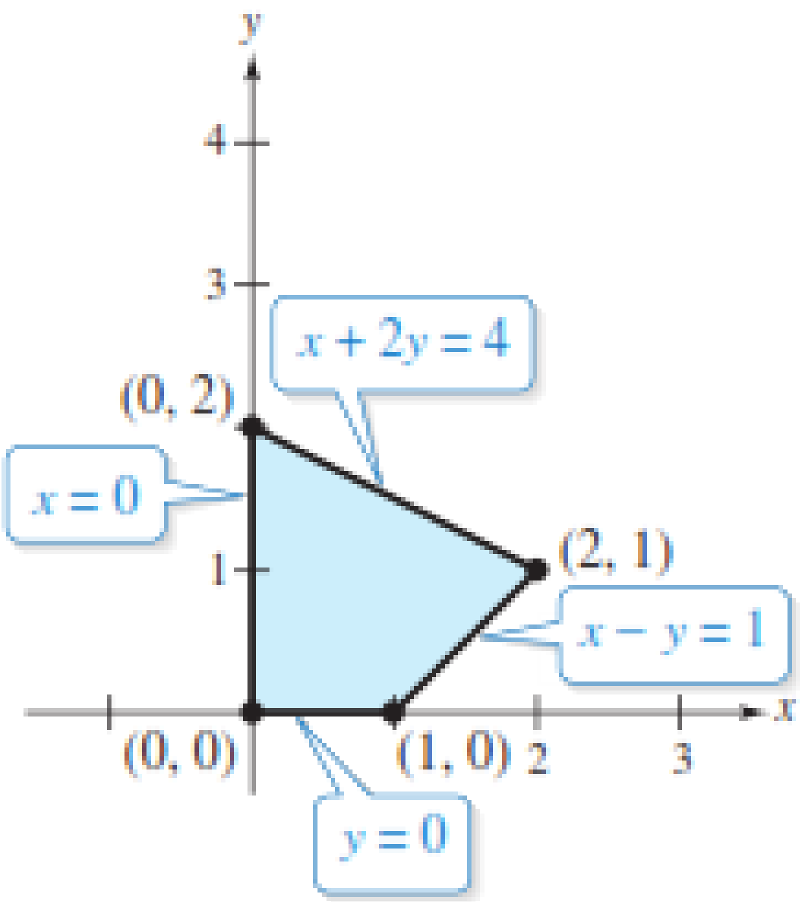
Find the maximum value of

$z = 3x + 2y$ Objective function

subject to the constraints listed below.

$$\left.\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned}\right\}$$ Constraints

Example (2,1)



Example

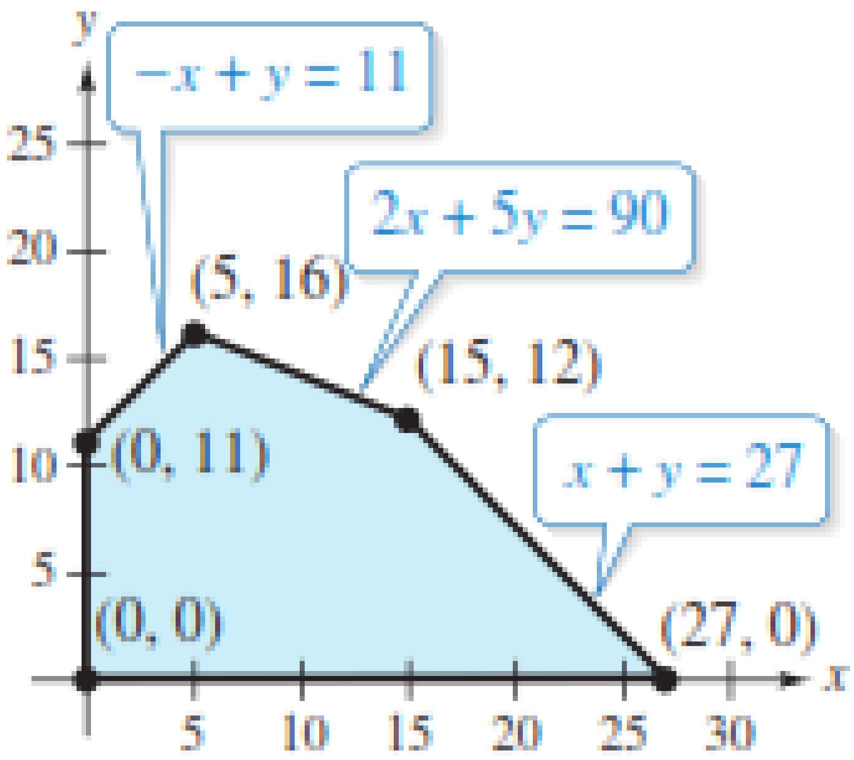
Find the maximum value of the objective function

$$z = 4x + 6y \qquad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left.\begin{aligned} -x + y &\leq 11 \\ x + y &\leq 27 \\ 2x + 5y &\leq 90. \end{aligned}\right\} \qquad \text{Constraints}$$

(15,12)


y
−x + y = 11
2x + 5y = 90
x + y = 27
(5, 16)
(15, 12)
(0, 11)
(0, 0)
(27, 0)
25
20
15
10
5
5
10
15
20
25
30
x

Example

2. A farmer has a 320 acre farm on which she plants two crops: corn and soybeans. For each acre of corn planted, her expenses are $50 and for each acre of soybeans planted, her expenses are $100. Each acre of corn requires 100 bushels of storage and yields a profit of $60. Each acre of soybeans requires 40 bushels of storage and yields a profit of $90. If the total amount of storage space available is 19,200 bushels and the farmer has only $20,000 on hand, how many acres of each crop should she plant in order to maximize her profit? What will her profit be if she follows this strategy?

Solution

x: acres of corn
y: acres of soybeans

$$P = 60x + 90y$$

$$\begin{cases} x + y \leq 320 & \text{(acres)} \\ 50x + 100y \leq 20{,}000 & \text{(expenses)} \\ 100x + 40y \leq 19{,}200 & \text{(storage)} \\ x \geq 0 \\ y \geq 0 \end{cases}$$

1. A potter is making cups and plates. It takes her 6 minutes to make a cup and 3 minutes to make a plate. Each cup uses ¾ lb of clay and each plate use one lb of clay. She has 20 hours available for making the cups and plates and has 250 lbs of clay on hand. She makes a profit of $2 on each cup and $1.50 on each plate. How many cups and how many plates should she make in order to maximize her profit?

x: # of cups
y: # of plates

$P = 2x + 1.50y$

$$\begin{cases} 6x + 3y \leq 1200 & \text{(time)} \\ \frac{3}{4}x + y \leq 250 & \text{(clay)} \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Application

LINEAR ALGEBRA APPLIED

When financial institutions replenish automatic teller machines (ATMs), they need to take into account a large number of variables and constraints to keep the machines stocked appropriately. Demand for cash machines fluctuates with such factors as weather, economic conditions, day of the week, and even road construction. Further complicating the matter in the United States is a penalty for depositing and withdrawing money from the Federal Reserve in the same week. To address this complex problem, a company that specializes in providing financial services technology can create high-end optimization software to set up and solve a linear programming problem with many variables and constraints. The company determines an equation for the objective function to minimize total cash in ATMs, while establishing constraints on travel routes, service vehicles, penalty fees, and so on. The optimal solution generated by the software allows the company to build detailed ATM restocking schedules.

Simplex Method

Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** when it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \cdot \cdot \cdot + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdot \cdot \cdot + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdot \cdot \cdot + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdot \cdot \cdot + a_{mn}x_n &\le b_m \end{aligned}$$

where $x_i \ge 0$ and $b_i \ge 0$. After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdot \cdot \cdot + a_{1n}x_n + s_1 \qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdot \cdot \cdot + a_{2n}x_n \qquad + s_2 \qquad &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdot \cdot \cdot + a_{mn}x_n \qquad\qquad + s_m &= b_m \end{aligned}$$

where $s_i \ge 0$.

Example

maximum value of $z = 4x_1 + 6x_2$,

$$
\begin{aligned}
-x_1 + \ \ x_2 &\le 11 \\
x_1 + \ \ x_2 &\le 27 \\
2x_1 + 5x_2 &\le 90.
\end{aligned}
$$

Slack Variables

$$
\begin{aligned}
-x_1 + x_2 + s_1 \qquad\qquad &= 11 \\
x_1 + x_2 \qquad + s_2 \qquad &= 27 \\
2x_1 + 5x_2 \qquad\qquad + s_3 &= 90.
\end{aligned}
$$

$$-c_1x_1 - c_2x_2 - \cdot \cdot \cdot - c_nx_n + (0)s_1 + (0)s_2 + \cdot \cdot \cdot + (0)s_m + z = 0.$$

slack variable" called an **artificial variable**

X1,x2 = Non basic variable

$$
\begin{aligned}
z &= 4x_1 + 6x_2 && \text{Objective function}\\
-x_1 + x_2 + s_1 &= 11\\
x_1 + x_2 + s_2 &= 27 && \text{Constraints}\\
2x_1 + 5x_2 + s_3 &= 90
\end{aligned}
$$

is shown below.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
−1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
−4	−6	0	0	0	0	

↑ Current z-value

Initial Solution

$$x_1 = 0$$
$$x_2 = 0$$
$$s_1 = 11$$
$$s_2 = 27$$
$$s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

Step 2

- To perform an **optimality check for a solution represented by a simplex tableau,**
- look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not optimal.*

Pivoting

1. The **entering variable** corresponds to the least (the most negative) entry in the bottom row of the tableau, excluding the "b-column."
2. The **departing variable** corresponds to the least nonnegative ratio b_i/a_{ij} in the column determined by the entering variable, when $a_{ij} > 0$.
3. The entry in the simplex tableau in the entering variable's column and the departing variable's row is the **pivot.**

Contd...

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
−1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
−4	−6	0	0	0	0	

SOLUTION

The objective function for this problem is

$$z = 4x_1 + 6x_2.$$

Note that the current solution

$$(x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90)$$

corresponds to a z-value of 0. To improve this solution, choose x_2 as the entering variable, because −6 is the least entry in the bottom row.

Contd…

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
−1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
−4	−6	0	0	0	0	
	↑ Entering					

To see *why* you choose x_2 as the entering variable, remember that $z = 4x_1 + 6x_2$. So, it appears that a unit change in x_2 produces a change of 6 in z, whereas a unit change in x_1 produces a change of only 4 in z.

Contd…

- In the event of a tie when choosing entering and/or departing variables, any of the tied variables may be chosen.

To find the departing variable, locate the b_i's that have corresponding positive elements in the entering variable's column and form the ratios

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \text{and} \quad \frac{90}{5} = 18.$$

Here the least nonnegative ratio is 11, so choose s_1 as the departing variable.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables	
−1	(1)	1	0	0	11	s_1	← Departing
1	1	0	1	0	27	s_2	
2	5	0	0	1	90	s_3	
−4	−6	0	0	0	0		
	↑ Entering						

Note that the pivot is the entry in the first row and second column. Now, use Gauss-Jordan elimination to obtain the improved solution shown below.

Before Pivoting

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

After Pivoting

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \begin{matrix} \\ -R_1 + R_2 \\ -5R_1 + R_3 \\ 6R_1 + R_4 \end{matrix}$$

The new tableau is shown below.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	66	

Note that x_2 has replaced s_1 in the basic variables column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a z-value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$

II Iteration

In Example 1, the improved solution is not optimal because the bottom row has a negative entry. So, apply another iteration of the simplex method to improve the solution further. Choose x_1 as the entering variable. Moreover, the lesser of the ratios $16/2 = 8$ and $35/7 = 5$ is 5, so s_3 is the departing variable. Gauss-Jordan elimination produces the matrices shown below.

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{7}R_3 \\ \\ \end{matrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & \frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ R_2 - 2R_3 \\ \\ R_4 + 10R_3 \end{matrix}$$

So, the new simplex tableau is as shown below.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

In this tableau, there is still a negative entry in the bottom row. So, choose s_1 as the entering variable and s_2 as the departing variable, as shown in the next tableau.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{5}{7}$	1	$-\frac{2}{7}$	6	s_2 ← Departing
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	
		↑ Entering				

One more iteration of the simplex method gives the tableau below. (Check this.)

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← Maximum z-value

In this tableau, there are no negative elements in the bottom row. So, the optimal solution is

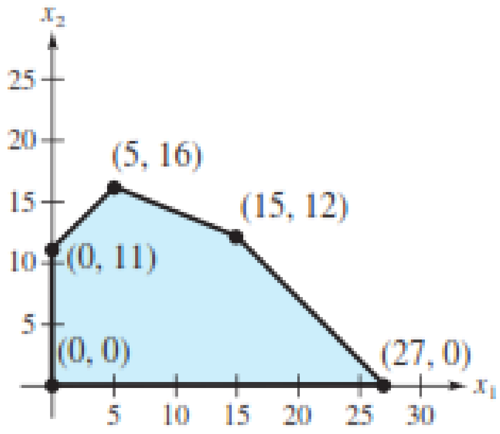
$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

$(0, 0)$ → $(0, 11)$ → $(5, 16)$ → $(15, 12)$

$z = 0$ → $z = 66$ → $z = 116$ → $z = 132$



The Simplex Method (Standard Form)

To solve a linear programming problem in standard form, use the steps below.

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row, excluding the "*b*-column." This entry is called the entering variable, and its column is the **entering column.** (If ties occur, then any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the "*b*-column" with their corresponding positive entries in the entering column. (If all entries in the entering column are 0 or negative, then there is no maximum solution.) The **departing row** corresponds to the least nonnegative ratio b_i/a_{ij}. (For ties, choose any corresponding row.) The entry in the departing row and the entering column is called the **pivot.**
5. Use elementary row operations to change the pivot to 1 and all other entries in the entering column to 0. This process is called **pivoting.**
6. When all entries in the bottom row are zero or positive, this is the final tableau. Otherwise, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution. The maximum value of the objective function is the entry in the lower right corner of the tableau.

$$\begin{aligned} \text{maximize} \quad & 3x + 5y \\ \text{subject to} \quad & x + y \le 4 \\ & x + 3y \le 6 \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

$$
\begin{array}{ll}
\text{maximize} & 3x + 5y + 0u + 0v \\
\text{subject to} & x + y + u = 4 \\
& x + 3y + v = 6 \\
& x, y, u, v \geq 0.
\end{array}
$$

(3,1) z=14

	x	y	u	v	
u	1	1	1	0	4
← v	1	3	0	1	6
	–3	–5	0	0	0
		↑			

(2,0) Z = 6

$$\begin{aligned} \text{maximize} \quad & 3x + y \\ \text{subject to} \quad & -x + y \le 1 \\ & 2x + y \le 4 \\ & x \ge 0,\ y \ge 0 \end{aligned}$$

	x	y	u	v	
u	−1	1	1	0	1
← v	2	1	0	1	4
	−3	−1	0	0	0
	↑				

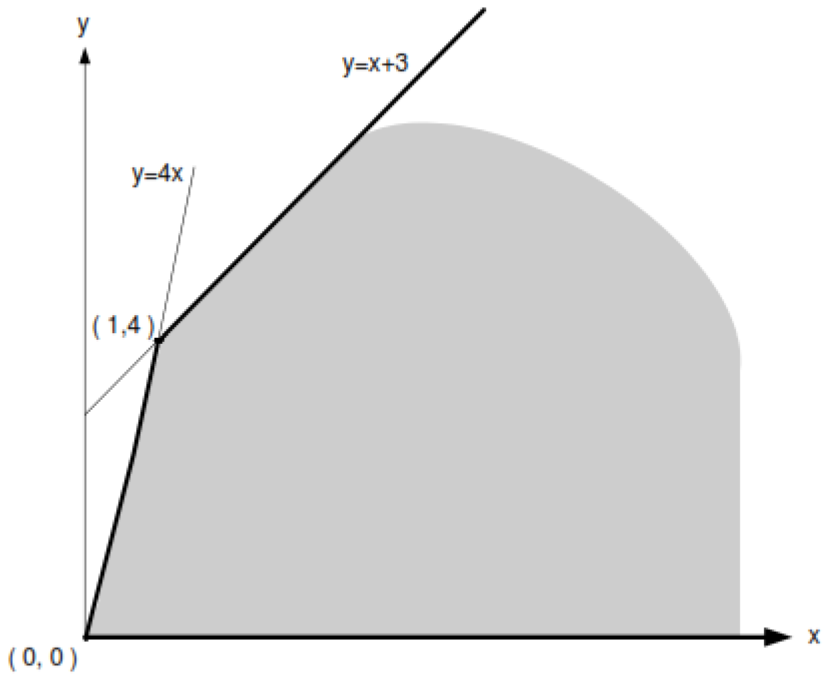
$\theta_v = \frac{4}{2}$

	x	y	u	v	
u	0	$\frac{3}{2}$	1	$\frac{1}{2}$	3
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	2
	0	$\frac{1}{2}$	0	$\frac{3}{2}$	6

The optimal solution found is $x = 2$, $y = 0$, with the maximal value of the objective function equal to 6.

$$\begin{aligned} \text{maximize} \quad & x + 2y \\ \text{subject to} \quad & 4x \geq y \\ & y \leq 3 + x \\ & x \geq 0,\ y \geq 0 \end{aligned}$$

Unbounded



$$
\begin{array}{ll}
\text{maximize} & x + 2y \\
\text{subject to} & -4x + y + u \qquad = 0 \\
 & -x + y + \qquad + v = 3 \\
 & x, y, u, v \geq 0.
\end{array}
$$

		x	y	u	v		
←	u	−4	1	1	0	0	$\theta_u = \frac{0}{1}$
	v	−1	1	0	1	3	$\theta_v = \frac{3}{1}$
		−1	−2	0	0	0	
			↑				

		x	y	u	v		
	y	−4	1	1	0	0	
←	v	3	0	−1	1	3	$\theta_v = \frac{3}{3}$
		−9	0	2	0	0	
		↑					

	x	y	u	v	
y	0	1	$-\frac{1}{3}$	$\frac{4}{3}$	4
x	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	1
	0	0	−1	3	9
			↑		

Find the dual of the linear programming problem

$$\begin{aligned} \text{maximize} \quad & x_1 + 4x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 \le 6 \\ & x_1 - x_2 - 2x_3 \le 2 \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Solution

$$
\begin{array}{rl}
\text{minimize} & 6y_1 + 2y_2 \\
\text{subject to} & y_1 + y_2 \geq 1 \\
& y_1 - y_2 \geq 4 \\
& y_1 - 2y_2 \geq -1 \\
& y_1, y_2 \geq 0.
\end{array}
$$

Example

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3$$ Objective function

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 \quad\quad &\leq 10 \\ x_1 + 2x_2 - 2x_3 &\leq 20 \\ x_2 + 2x_3 &\leq 5 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial and subsequent simplex tableaus for this problem are shown below. (Check the computations, and note the "tie" that occurs when choosing the first entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	
1	2	−2	0	1	0	20	s_2	
0	1	(2)	0	0	1	5	s_3	← Departing
−2	1	−2	0	0	0	0		
		↑ Entering						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
(2)	1	0	1	0	0	10	s_1	← Departing
1	3	0	0	1	1	25	s_2	
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3	
−2	2	0	0	0	1	5		
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	x_1
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
0	3	0	1	0	1	15	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = \left(5, 0, \tfrac{5}{2}, 0, 20, 0\right)$$

and the maximum value of z is 15.

Example

Use the simplex method to find the maximum value of

$$z = 3x_1 + 2x_2 + x_3$$ Objective function

subject to the constraints

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 30 \\ 2x_1 + 3x_2 + x_3 &\leq 60 \\ x_1 + 2x_2 + 3x_3 &\leq 40 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

Using the basic feasible solution $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 30, 60, 40)$, the initial and subsequent simplex tableaus for this problem are shown below. (Note that s_1 is an artificial variable, rather than a slack variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
(4)	1	1	1	0	0	30	s_1	← Departing
2	3	1	0	1	0	60	s_2	
1	2	3	0	0	1	40	s_3	
−3	−2	−1	0	0	0	0		
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{15}{2}$	x_1
0	$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	45	s_2 ← Departing
0	$\frac{7}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	0	1	$\frac{65}{2}$	s_3
0	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{45}{2}$	
	↑ Entering						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	0	$\frac{1}{5}$	$\frac{3}{10}$	$-\frac{1}{10}$	0	3	x_1
0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	0	18	x_2
0	0	$\frac{12}{5}$	$\frac{1}{10}$	$-\frac{7}{10}$	1	1	s_3
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	45	

So the optimal solution is $(x_1, x_2, x_3, s_1, s_2, s_3) = (3, 18, 0, 0, 0, 1)$ and the maximum value of z is 45. This solution satisfies the equation provided in the constraints, because $4(3) + 1(18) + 1(0) = 30$. ■

A manufacturer produces three types of plastic fixtures. The table below shows the times required for molding, trimming, and packaging. (Times are in hours per dozen fixtures, and profits are in dollars per dozen fixtures.)

Process	*Type A*	*Type B*	*Type C*
Molding	1	2	$\frac{3}{2}$
Trimming	$\frac{2}{3}$	$\frac{2}{3}$	1
Packaging	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
Profit	$11	$16	$15

The maximum amounts of production time that the manufacturer can allocate to each component are listed below.

Molding: 12,000 hours

Trimming: 4600 hours

Packaging: 2400 hours

How many dozen units of each type of fixture should the manufacturer produce to obtain a maximum profit?

Solution

Let x_1, x_2, and x_3 represent the numbers of dozens of types A, B, and C fixtures, respectively. The objective function to be maximized is

$$\text{Profit} = 11x_1 + 16x_2 + 15x_3$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Moreover, using the information in the table, you can write the constraints below.

$$\begin{aligned} x_1 + 2x_2 + \tfrac{3}{2}x_3 &\leq 12{,}000 \\ \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + x_3 &\leq 4600 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &\leq 2400 \end{aligned}$$

So, the initial simplex tableau with the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12{,}000, 4600, 2400)$$

is as shown below.

A minimization problem is in **standard form** when the objective function

$$w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

Minimisation problem

Minimization Problem: Find the minimum value of

$$w = 0.12x_1 + 0.15x_2 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 60x_1 + 60x_2 &\geq 300 \\ 12x_1 + 6x_2 &\geq 36 \\ 10x_1 + 30x_2 &\geq 90 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

$$\begin{bmatrix} 60 & 60 & 300 \\ 12 & 6 & 36 \\ 10 & 30 & 90 \\ 0.12 & 0.15 & 0 \end{bmatrix}$$

Next, form the transpose of this matrix.

$$\begin{bmatrix} 60 & 12 & 10 & 0.12 \\ 60 & 6 & 30 & 0.15 \\ 300 & 36 & 90 & 0 \end{bmatrix}$$

To interpret the transposed matrix as a maximization problem, introduce new variables, y_1, y_2, and y_3. This corresponding maximization problem is called the **dual** of the original minimization problem.

Dual Maximization Problem: Find the maximum value of

$$z = 300y_1 + 36y_2 + 90y_3$$ Dual objective function

subject to the constraints

$$\left.\begin{aligned} 60y_1 + 12y_2 + 10y_3 &\le 0.12 \\ 60y_1 + 6y_2 + 30y_3 &\le 0.15 \end{aligned}\right\}$$ Constaints

where $y_1 \ge 0$, $y_2 \ge 0$, and $y_3 \ge 0$.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
(60)	12	10	1	0	0.12	s_1	← Departing
60	6	30	0	1	0.15	s_2	
−300	−36	−90	0	0	0		
↑ Entering							

y_1	y_2	y_3	s_1	s_2	b	Basic Variables	
1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{60}$	0	$\frac{1}{500}$	y_1	
0	−6	(20)	−1	1	$\frac{3}{100}$	s_2	← Departing
0	24	−40	5	0	$\frac{3}{5}$		
		↑ Entering					

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{4}$	0	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{7}{4000}$	y_1
0	$-\frac{3}{10}$	1	$-\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{2000}$	y_3
0	12	0	3	2	$\frac{33}{50}$	
			↑ x_1	↑ x_2		

THEOREM 9.2 The von Neumann Duality Principle

The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z.

Summary

Solving a Minimization Problem

A minimization problem is in standard form when the objective function

$$w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where

$$x_i \geq 0 \quad \text{and} \quad b_i \geq 0.$$

To solve this problem, use the steps below.

1. Form the **augmented matrix** for the system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \\ c_1 & c_2 & \ldots & c_n & 0 \end{bmatrix}$$

2. Form the transpose of this matrix.

$$\begin{bmatrix} a_{11} & a_{21} & \ldots & a_{m1} & c_1 \\ a_{12} & a_{22} & \ldots & a_{m2} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{mn} & c_n \\ b_1 & b_2 & \ldots & b_m & 0 \end{bmatrix}$$

3. Form the **dual maximization problem** corresponding to this transposed matrix. That is, find the maximum of the objective function

$$z = b_1y_1 + b_2y_2 + \cdot \cdot \cdot + b_my_m$$

subject to the constraints

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdot \cdot \cdot + a_{m1}y_m &\le c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdot \cdot \cdot + a_{m2}y_m &\le c_2 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdot \cdot \cdot + a_{mn}y_m &\le c_n \end{aligned}$$

where

$$y_1 \ge 0, \quad y_2 \ge 0, \quad \ldots, \quad \text{and} \quad y_m \ge 0.$$

Apply the **simplex method** to the dual maximization problem. The maximum value of z will be the minimum value of w. Moreover, the values of $x_1, x_2, \ldots, x_n$ will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

Example

Find the minimum value of

$$w = 3x_1 + 2x_2$$ Objective function

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + x_2 &\geq 4 \end{aligned}\right\}$$ Constraints

where $x_1 \geq 0$ and $x_2 \geq 0$.

The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 4 \\ 3 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}$$

Minimization Problem | Dual Maximization Problem

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 4y_2$$ Dual objective function

subject to the constraints

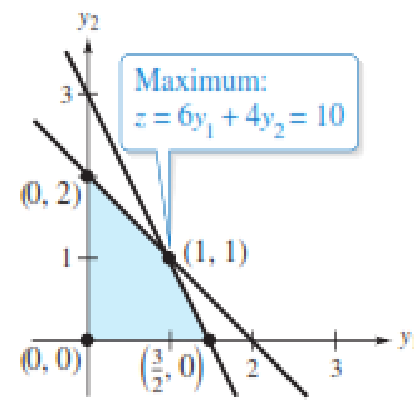
$$\left.\begin{aligned} 2y_1 + y_2 &\le 3 \\ y_1 + y_2 &\le 2 \end{aligned}\right\}$$ Dual constraints

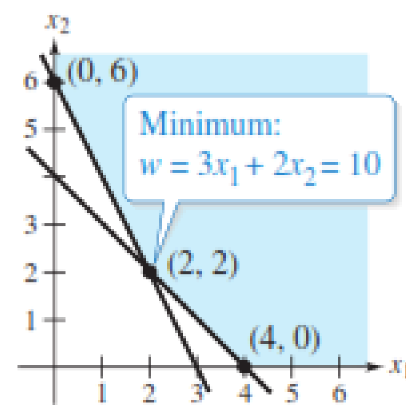
where $y_1 \ge 0$ and $y_2 \ge 0$. Now apply the simplex method to the dual problem, as shown below.

y_1	y_2	s_1	s_2	b	Basic Variables	
2	1	1	0	3	s_1	← Departing
1	1	0	1	2	s_2	
−6	−4	0	0	0		
↑ Entering						

y_1	y_2	s_1	s_2	b	Basic Variables	
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	y_1	
0	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	s_2	← Departing
0	−1	3	0	9		
	↑ Entering					

y_1	y_2	s_1	s_2	b	Basic Variables
1	0	1	−1	1	y_1
0	1	−1	2	1	y_2
0	0	2	2	10	
		↑ x_1	↑ x_2		





Example

Find the minimum value of $w = 2x_1 + 10x_2 + 8x_3$ subject to the constraints

$$\left.\begin{aligned} x_1 + x_2 + x_3 &\geq 6 \\ x_2 + 2x_3 &\geq 8 \\ -x_1 + 2x_2 + 2x_3 &\geq 4 \end{aligned}\right\} \quad \text{Constraints}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ -1 & 2 & 2 & 4 \\ 2 & 10 & 8 & 0 \end{bmatrix}$$

Minimization Problem

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 2 & 10 \\ 1 & 2 & 2 & 8 \\ 6 & 8 & 4 & 0 \end{bmatrix}$$

Dual Maximization Problem

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 8y_2 + 4y_3 \qquad \text{Dual objective function}$$

subject to the constraints

$$\left.\begin{aligned} y_1 \qquad\quad - \ y_3 &\le 2 \\ y_1 + \ y_2 + 2y_3 &\le 10 \\ y_1 + 2y_2 + 2y_3 &\le 8 \end{aligned}\right\} \qquad \text{Dual constraints}$$

where $y_1 \ge 0$, $y_2 \ge 0$, and $y_3 \ge 0$. Now apply the simplex method to the dual problem as shown below.

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables	
1	0	−1	1	0	0	2	s_1	
1	1	2	0	1	0	10	s_2	
1	(2)	2	0	0	1	8	s_3	← Departing
−6	−8	−4	0	0	0	0		
	↑ Entering							

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables	
(1)	0	−1	1	0	0	2	s_1	← Departing
$\frac{1}{2}$	0	1	0	1	$-\frac{1}{2}$	6	s_2	
$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	4	y_2	
−2	0	4	0	0	4	32		
↑ Entering								

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables
1	0	-1	1	0	0	2	y_1
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	5	s_2
0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	y_2
0	0	2	2	0	4	36	
			↑	↑	↑		
			x_1	x_2	x_3		

From this final simplex tableau, the maximum value of z is 36. So, the minmum value of w is 36, and this occurs when $x_1 = 2$, $x_2 = 0$, and $x_3 = 4$.

Example

A petroleum company owns two refineries. Refinery 1 costs $20,000 per day to operate, and refinery 2 costs $25,000 per day to operate. The table shows the numbers of barrels of each grade of oil the refineries can produce each day.

Grade	*Refinery 1*	*Refinery 2*
High-grade	400	300
Medium-grade	300	400
Low-grade	200	500

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

Let x_1 and x_2 represent the numbers of days the two refineries operate. Then the total cost is

$$C = 20{,}000x_1 + 25{,}000x_2. \qquad \text{Objective function}$$

The constraints are

$$\left.\begin{aligned} \text{(High-grade)} \quad 400x_1 + 300x_2 &\geq 25{,}000 \\ \text{(Medium-grade)} \quad 300x_1 + 400x_2 &\geq 27{,}000 \\ \text{(Low-grade)} \quad 200x_1 + 500x_2 &\geq 30{,}000 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 400 & 300 & 25{,}000 \\ 300 & 400 & 27{,}000 \\ 200 & 500 & 30{,}000 \\ 20{,}000 & 25{,}000 & 0 \end{bmatrix}$$

Minimization Problem

$$\begin{bmatrix} 400 & 300 & 200 & 20{,}000 \\ 300 & 400 & 500 & 25{,}000 \\ 25{,}000 & 27{,}000 & 30{,}000 & 0 \end{bmatrix}$$

Dual Maximization Problem

Now apply the simplex method to the dual problem.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
400	300	200	1	0	20,000	s_1
300	400	(500)	0	1	25,000	s_2 ← Departing
−25,000	−27,000	−30,000	0	0	0	
		↑ Entering				

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
(280)	140	0	1	$-\frac{2}{5}$	10,000	s_1 ← Departing
$\frac{3}{5}$	$\frac{4}{5}$	1	0	$\frac{1}{500}$	50	y_3
−7000	−3000	0	0	60	1,500,000	
↑ Entering						

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	y_1
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	y_3
0	500	0	25	50	1,750,000	
			↑ x_1	↑ x_2		

From this final simplex tableau, the minimum cost is

$C = \$1{,}750{,}000$ Minimum cost

and this occurs when

$x_1 = 25$ and $x_2 = 50.$

So, the two refineries should be operated for the numbers of days shown below.

Refinery 1: 25 days

Refinery 2: 50 days

Note that by operating the two refineries for these numbers of days, the company produces the amounts of oil listed below.

High-grade oil: $400(25) + 300(50) = 25{,}000$ barrels
Medium-grade oil: $300(25) + 400(50) = 27{,}500$ barrels
Low-grade oil: $200(25) + 500(50) = 30{,}000$ barrels

So, the company refines enough of each grade of oil to meet its orders (with a surplus of 500 barrels of medium-grade oil).

Mixed-Constraint Problem: Find the maximum value of

$$z = x_1 + x_2 + 2x_3$$ Objective function

subject to the constraints

$$\left.\begin{aligned} 2x_1 + x_2 + x_3 &\le 50 \\ 2x_1 + x_2 \quad\quad &\ge 36 \\ x_1 \quad\quad + x_3 &\ge 10 \end{aligned}\right\}$$ Constraints

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$. This is a maximization problem,

$$2x_1 + x_2 + x_3 + s_1 = 50.$$

For the other two inequalities, a new type of variable, a **surplus variable,** is introduced, as shown below.

$$\begin{aligned} 2x_1 + x_2 \quad\quad - s_2 \quad\quad &= 36 \\ x_1 \quad\quad + x_3 \quad\quad - s_3 &= 10 \end{aligned}$$

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
2	1	1	1	0	0	50	s_1
2	1	0	0	−1	0	36	s_2
1	0	1	0	0	−1	10	s_3
−1	−1	−2	0	0	0	0	

To eliminate the surplus variables from the current solution, use "trial and error." That is, in an effort to find a feasible solution, arbitrarily choose new entering variables. For example, it seems reasonable to select x_3 as the entering variable, because its column has the most negative entry in the bottom row.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	1	1	0	0	50	s_1	
2	1	0	0	−1	0	36	s_2	
1	0	(1)	0	0	−1	10	s_3	← Departing
−1	−1	−2	0	0	0	0		
		↑ Entering						

After pivoting, the new simplex tableau is as shown below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	1	0	1	0	1	40	s_1	
2	(1)	0	0	−1	0	36	s_2	← Departing
1	0	1	0	0	−1	10	x_3	
1	−1	0	0	0	−2	20		
	↑ Entering							

The current solution $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 10, 40, -36, 0)$ is still not feasible, so choose x_2 as the entering variable and pivot to obtain the simplex tableau below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
-1	0	0	1	1	(1)	4	s_1	← Departing
2	1	0	0	-1	0	36	x_2	
1	0	1	0	0	-1	10	x_3	
3	0	0	0	-1	-2	56		
					↑ Entering			

At this point, you obtain a feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 36, 10, 4, 0, 0).$$

From here, continue by applying the simplex method as usual. Note that the next entering variable is s_3. After pivoting one more time, you obtain the final simplex tableau shown below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-1	0	0	1	1	1	4	s_3
2	1	0	0	-1	0	36	x_2
0	0	1	1	1	0	14	x_3
1	0	0	2	1	0	64	

Note that this tableau is final because it represents a feasible solution and there are no negative entries in the bottom row. So, the maximum value of the objective function is $z = 64$ and this occurs when $x_1 = 0$, $x_2 = 36$, and $x_3 = 14$.

Find the maximum value of

$$z = 3x_1 + 2x_2 + 4x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\left.\begin{aligned} 3x_1 + 2x_2 + 5x_3 &\le 18 \\ 4x_1 + 2x_2 + 3x_3 &\le 16 \\ 2x_1 + x_2 + x_3 &\ge 4 \end{aligned}\right\} \qquad \text{Constraints}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

$$
\begin{aligned}
3x_1 + 2x_2 + 5x_3 + s_1 \qquad\qquad &= 18 \\
4x_1 + 2x_2 + 3x_3 \qquad + s_2 \qquad &= 16 \\
2x_1 + x_2 + x_3 \qquad\qquad - s_3 &= 4
\end{aligned}
$$

Next form the initial simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
3	2	5	1	0	0	18	s_1	
4	2	3	0	1	0	16	s_2	
2	1	1	0	0	−1	4	s_3	← Departing
−3	−2	−4	0	0	0	0		

This tableau does not represent a feasible solution because the value of s_3 is negative. So, s_3 should be the departing variable. There are no real guidelines as to which variable should enter the solution, and in fact, any choice will work. However, some entering variables will require more tedious computations than others. For example, choosing x_1 as the entering variable produces the tableau below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	$\frac{1}{2}$	$\frac{7}{2}$	1	0	$\frac{3}{2}$	12	s_1
0	0	1	0	1	2	8	s_2
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	2	x_1
0	$-\frac{1}{2}$	$-\frac{5}{2}$	0	0	$-\frac{3}{2}$	6	

Choosing x_2 as the entering variable on the initial tableau instead produces the tableau shown below, which contains "nicer" numbers.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
−1	0	3	1	0	2	10	s_1
0	0	1	0	1	2	8	s_2
2	1	1	0	0	−1	4	x_2
1	0	−2	0	0	−2	8	

Choosing x_3 as the entering variable on the initial tableau will also produce a tableau that does not contain fractions. (Verify this.)

X2 entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
-1	0	(3)	1	0	2	10	s_1	← Departing
0	0	1	0	1	2	8	s_2	
2	1	1	0	0	-1	4	x_2	
1	0	-2	0	0	-2	8		
		↑ Entering						

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
$-\frac{1}{3}$	0	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{10}{3}$	x_3	
$\frac{1}{3}$	0	0	$-\frac{1}{3}$	1	($\frac{4}{3}$)	$\frac{14}{3}$	s_2	← Departing
$\frac{7}{3}$	1	0	$-\frac{1}{3}$	0	$-\frac{5}{3}$	$\frac{2}{3}$	x_2	
$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{44}{3}$		
					↑ Entering			

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	x_3
$\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{7}{2}$	s_3
$\frac{11}{4}$	1	0	$-\frac{3}{4}$	$\frac{5}{4}$	0	$\frac{13}{2}$	x_2
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	17	

So, the maximum value of the objective function is $z = 17$, and this occurs when

$$x_1 = 0,\ x_2 = \tfrac{13}{2},\ \text{and}\ x_3 = 1.$$