Simplex Algorithm

Operations Research

- Operation Research is also called OR for short and it is a scientific approach to decision making which seeks to determine how best to design and operate a system under conditions requiring allocation of scarce resources.
- Operations research as a field, primarily has a set or collection of algorithms which act as tools for problems solving in chosen application areas.
- OR has extensive applications in engineering business and public systems and is also used by manufacturing and service industries to solve their day to day problems.

Contd…

- The history of the OR as a field as goes up to the Second World War. In fact this field operations research started during the Second World War when the British military asked scientists to analyze military problems.
- In fact Second World War was perhaps the first time when people realized that resources were scarce and had to be used effectively and allocated efficiently. The application of mathematics and scientific methods to military applications was called operations research to begin with.
- But today it has a different definition it is also called management science.

Contd…

- Linear programming was first conceived by Dantzig, around 1947 at the end of the Second World War.
- Very historically, the work of a Russian mathematician first had taken place in 1939 but since it was published in 1959,
- Dantzig was still credited with starting linear programming. In fact Dantzig did not use the term linear programming. His first paper was titled 'Programming in Linear Structure'.
- Much later, the term 'Linear Programming' was coined by Koopmans.
- The Simplex method which is the most popular and powerful tool to solve linear programming problems, was published by Dantzig in 1949.

- Two researchers, L. V. Kantorovich of the former Soviet Union and the Dutch-American T. C. Koopmans, were even awarded the Nobel Prize in 1975 for their contributions to linear programming theory and its applications to economics.
- G. B. Dantzig is the father of LP.

In an *optimization problem* one seeks to maximize or minimize a specific quantity, called the *objective*, which depends on a finite number of input variables. These variables may be independent of one another, or they may be related through one or more *constraints*.

$$
\begin{aligned}
&\text{optimize:} \quad z = f(x_1, x_2, \ldots, x_n) \\
&\text{subject to:} \quad
\left.\begin{matrix}
g_1(x_1, x_2, \ldots, x_n) \\
g_2(x_1, x_2, \ldots, x_n) \\
\cdots\cdots\cdots\cdots\cdots \\
\\
g_m(x_1, x_2, \ldots, x_n)
\end{matrix}\right\}
\begin{matrix} \le \\ = \\ \ge \end{matrix}
\left\{\begin{matrix}
b_1 \\
b_2 \\
\cdots \\
\\
b_m
\end{matrix}\right.
\end{aligned}
$$

$$f(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$g_i(x_1, x_2, \ldots, x_n) = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$$

Optimisation Problem

- Shortest path
- Minimum spanning tree

Iterative Improvement

Algorithm design technique for solving optimization problems

- Start with a feasible solution
- Repeat the following step until no improvement can be found:
  - change the current feasible solution to a feasible solution with a better value of the objective function
- Return the last feasible solution as optimal

Note: Typically, a change in a current solution is “small” (*local search*)

Major difficulty: Local optimum vs. global optimum

Linear programming (LP) problem is to optimize a linear function of several variables subject to linear constraints:

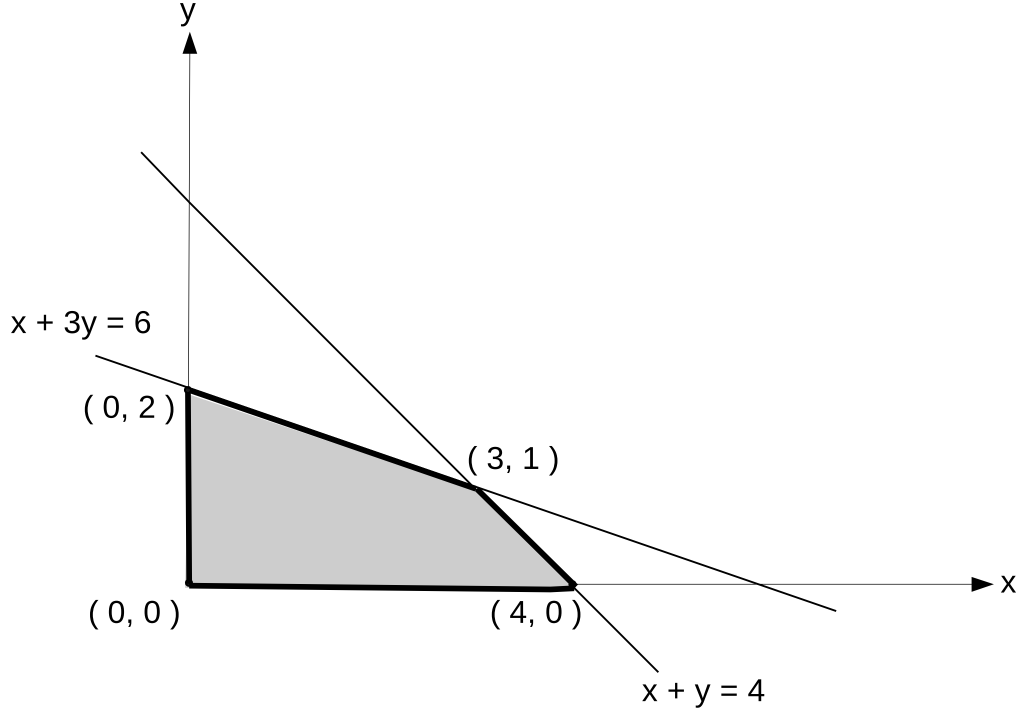
maximize (or minimize) $c_1 x_1 + ...+ c_n x_n$
subject to $a_{i\,1}x_1 + ...+ a_{in} x_n \leq$ (or $\geq$ or $=$) b_i, $i = 1,...,m$
$x_1 \geq 0, ... , x_n \geq 0$

The function $z = c_1 x_1 + ...+ c_n x_n$ is called the *objective function*;

constraints $x_1 \geq 0, ... , x_n \geq 0$ are called *nonnegativity constraints*

$$
\begin{aligned}
\text{maximize} \quad & 3x + 5y \\
\text{subject to} \quad & x + y \le 4 \\
& x + 3y \le 6 \\
& x \ge 0, \ y \ge 0
\end{aligned}
$$

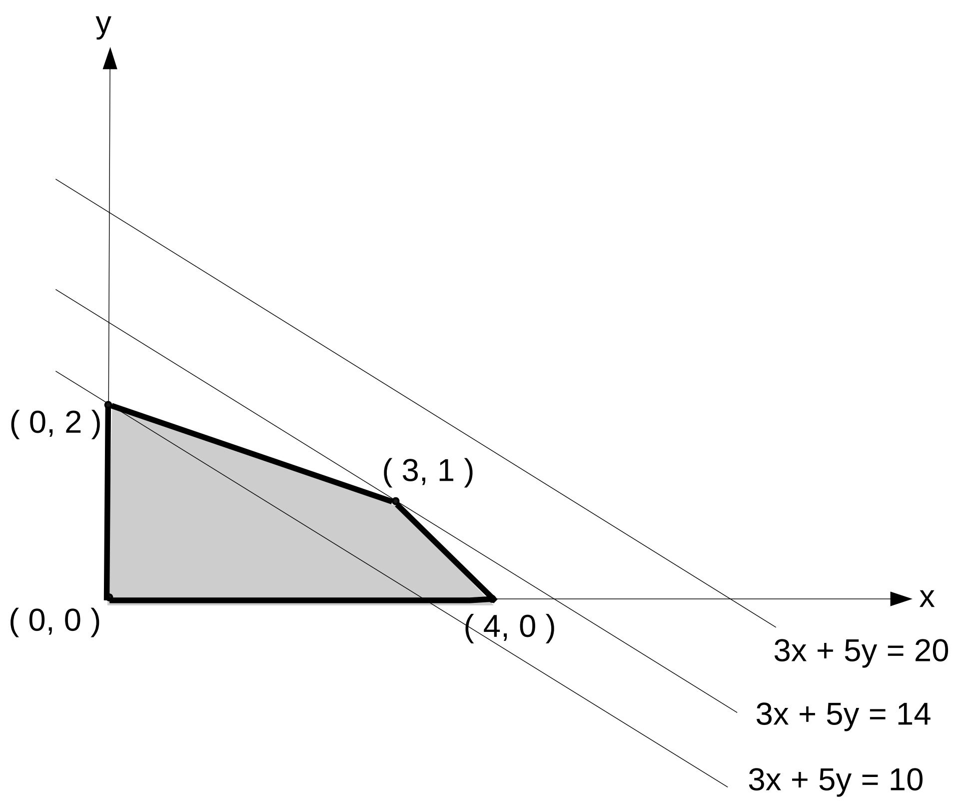
Feasible region



$$\begin{aligned} \text{maximize} \quad & 3x + 5y \\ \text{subject to} \quad & x + y \le 4 \\ & x + 3y \le 6 \\ & x \ge 0, \; y \ge 0 \end{aligned}$$

Optimal solution: $x = 3$, $y = 1$

Geometric solution



Example

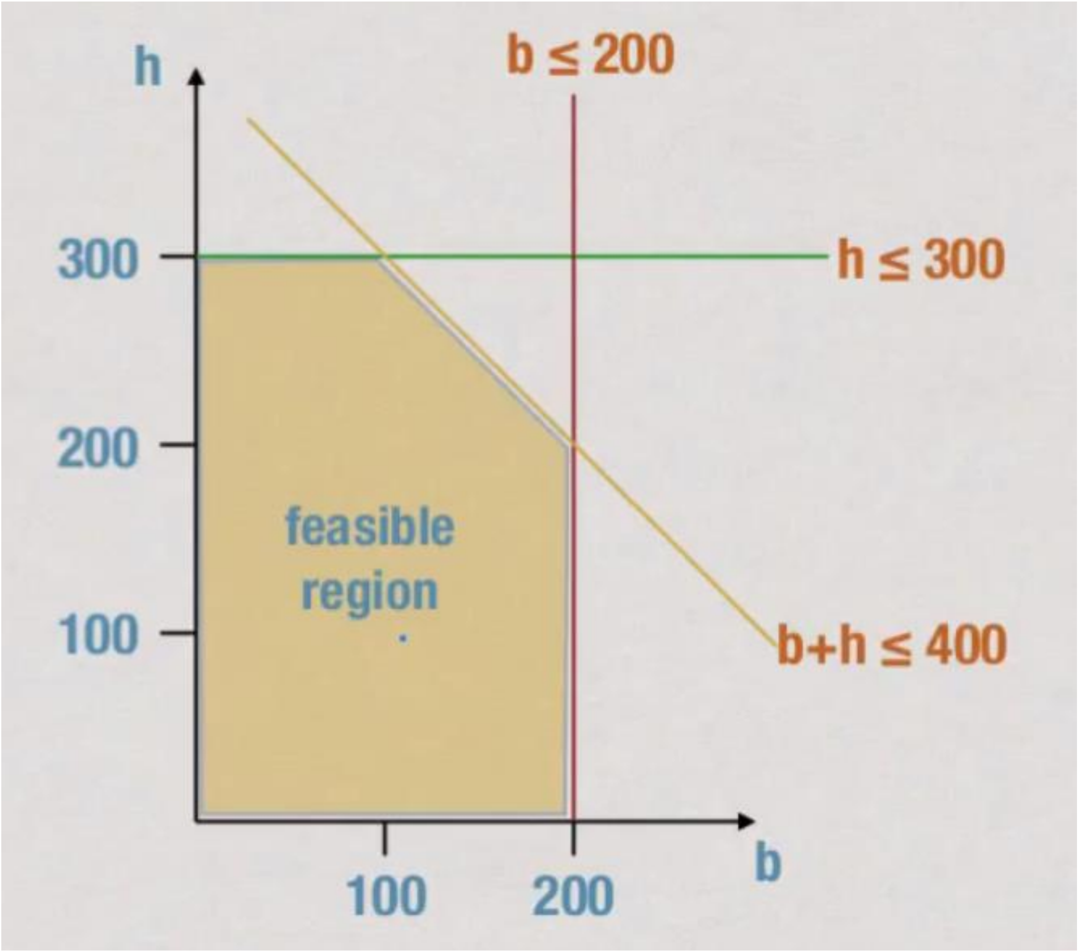
- Sweet shop sells Burfis and Halwas.
- Each box of burfi earns a profit of Rs.100/- and Halwa 600.
- Daily demand for burfi is atmost 200 and halwa is 300.
- Staff can produce 400 boxes a day altogether.
- What is the profitable mix of burfis and halwas to produce?

LP model

- Let b = no. of burfi boxes produced in a day
- Let h = no. of halwa boxes produced in a day
- Profit $z = 100b + 600h$
- Demand constraint: $b \leq 200$, $h \leq 300$
- Production constraint: $b + h \leq 400$
- Implicit constraints: $b, h \geq 0$

Contd…

- Objective function:
- $Z = 100b + 600h$
- Constraints:
- $b \leq 200$,
- $h \leq 300$
- $b + h \leq 400$
- $b, h \geq 0$



h
b ≤ 200
300
h ≤ 300
200
feasible
region
100
b+h ≤ 400
100
200
b

- Optimal value always occurs at the end point vertex.
- Solution:
  - Feasible solution is convex
  - Empty, constraints not satisfiable
  - May be unbounded, no upperlimit

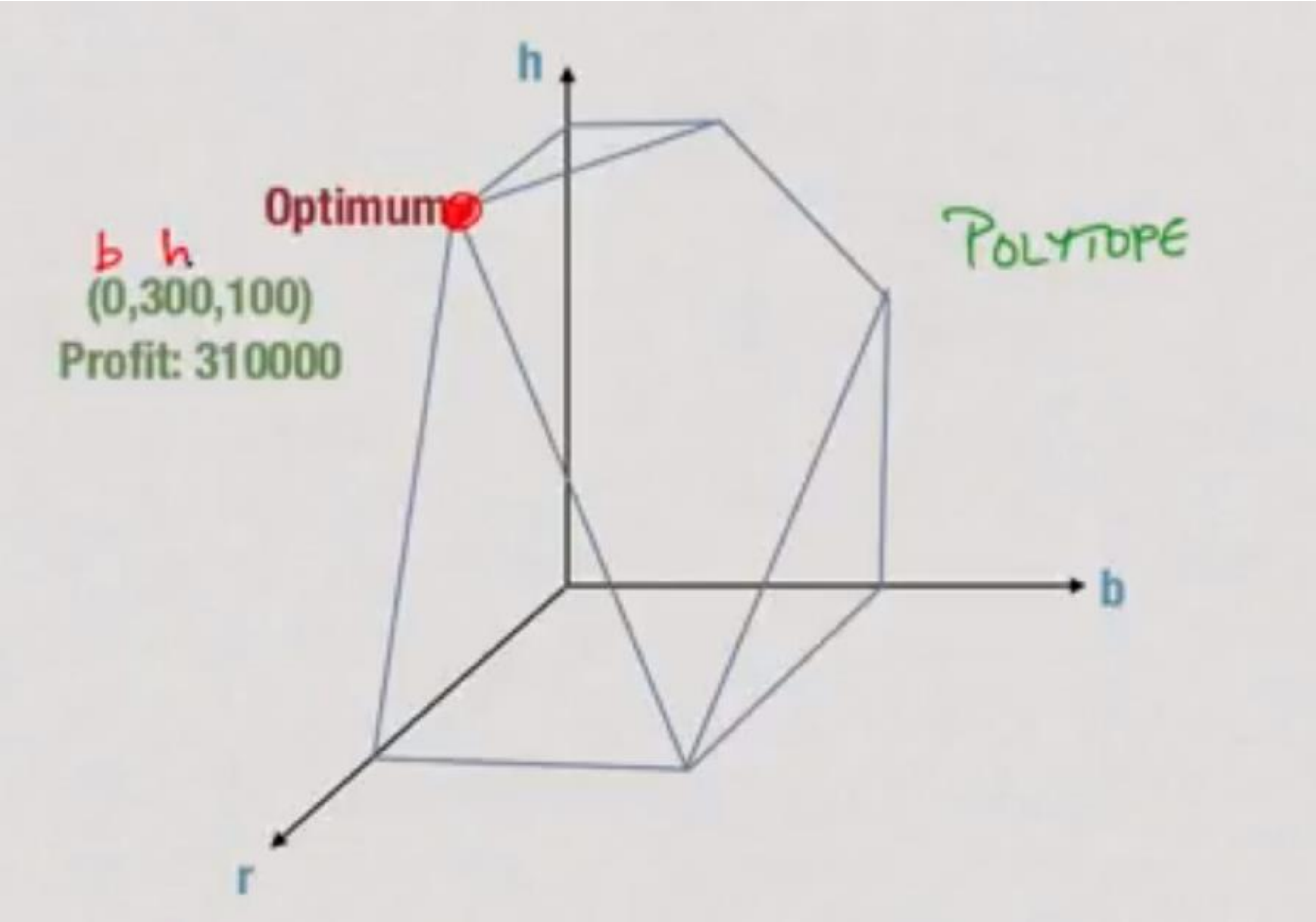
Simplex algorithm

- Start with any vertex, evaluate objective function
- If the adjacent vertex has better value, move.
- If the current vertex has better value than neighbours, stop.
- Can be exponential and efficient in practice.

- Narendra Karmarkar invented a polynomial algorithm for linear programming also known as the interior point method

- Sweet shop sells Burfis and Halwas. It adds rasamalai
- Each box of burfi earns a profit of Rs.100/- and Halwa 600, rasamalai 1300.
- Daily demand for burfi is atmost 200 and halwa is 300, rasamalai unlimited.
- Staff can produce 400 boxes a day altogether.
- Milk supply:600 box of halwa or 200 box of rasamalai or any combination.
- What is the profitable mix of burfis and halwas to produce?

- Objective function:
- $Z = 100b + 600h + 1300r$
- Constraints:
- $b \leq 200$,
- $h \leq 300$
- $b + h + r \leq 400$
- $h + 3r \leq 600$
- $b, h, r \geq 0$


h
Optimum
b h
(0,300,100)
Profit: 310000
POLYTOPE
b
r

Contd…

- Dual LP:
- $h \leq 300$ * 100
- $b+h + r \leq 400$ * 100
- $h+3r \leq 600$ * 400

Example

- For example we have these two products A and B. Two resources are need R one and R2. A requires one unit of R1 and three units of R two. B requires one unit of R one and two units of R two. Manufacturer has 5 units of R one available and 12 units of R two available. The manufacturer also makes a profit of rupees 6 per unit of A sold and rupees 5 per unit of B of sold. So this is the problem setting that we are looking at.

- let X = number of units of A produced
- Y = number of units of B produced.
- $Z = 6X + 5Y$.
- Constraints: $X + Y \leq 5$
- $3X + 2Y \leq 12$
- $X, Y \geq 0$

Production planning

- Let us consider a company making a single product demand of 1000, 800, 1200 and 900 respectively for 4 months. Now the company wants to meet the demand for the product in the next 4 months. The company can use two modes of production. There is something called as Regular and overtime production. Now the regular time capacity is 800/month and overtime capacity is 200/ month. In order to produce one item in regular time, it costs Rs 20 and to produce overtime it costs Rs 25. The company can also produce more in a particular month and carry the excess to the next month. Such a carrying cost is rupees 3.

- Let Xj. as quantity produced using regular time production in month j
- Yj as quantity produce using overtime in month j.
- Now these are our decision variables, Xj and Yj.
- I1 which is the quantity that is carried to the next month.

- Minimize:
- $Z = 20 (X1+X2+X3+X4) + 25(Y1+Y2+Y3+Y4) + 3(I1+I2+I3)$.
- $X1 + Y1 = 1000 + I1$
- $I1 +X2 + Y2 = 800 + I2$
- $I2 + X3 + Y3 = 1200 + I3$
- $I3 + X4 + y4 = 900$

Agenda

- Linear Programming problem
- Simplex – Algebraic form
- Worth of resource and dual
- Tabular form
- Matrix form
- Minimization/ $\geq$ constraint (dual simplex)
- Limitations – Cycling
- Computational issues - complexity

Simplex algorithm is used to solve Linear Programming problems (LPP)

What is a Linear Programming problem?

Example

A company can make 3 products. Sale price = 250, 180, 300. They are made on two machines – 10, 12, 15 and 24, 14, 20 on M1 and M2. Availability = 2400 and 4800. How much do they produce?

Let X_1, X_2, X_3 be the units of P, Q, R made

Decision variable

Maximize $250X_1 + 180X_2 + 300X_3$

Objective function

$10X_1 + 12X_2 + 15X_3 \leq 2400$

$24X_1 + 14X_2 + 20X_3 \leq 4800$

constraints

Linear

$X_1, X_2 \geq 0$

Non negativity

$$\text{Maximize } 250X_1 + 180X_2 + 300X_3$$
$$10X_1 + 12X_2 + 15X_3 \le 2400$$
$$24X_1 + 14X_2 + 20X_3 \le 4800$$
$$X_1, X_2 \ge 0$$

$$\text{Maximize } 250X_1 + 180X_2 + 300X_3 + 0X_4 + 0X_5$$
$$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$$
$$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$$
$$X_1, X_2 \ge 0$$

$$X_4 = 2400 - 10X_1 - 12X_2 - 15X_3$$
$$X_5 = 4800 - 24X_1 - 14X_2 - 20X_3$$
$$Z = 250X_1 + 180X_2 + 300X_3 + 0X_4 + 0X_5$$

$X_4 = 2400$, $X_5 = 4800$ $Z = 0$

X_3 comes into the solution.

Best value = min{2400/15, 4800/20} = min {160, 240} = 160

X_3 replaces X_4.

$X_4 = 2400 - 10X_1 - 12X_2 - 15X_3$

$X_5 = 4800 - 24X_1 - 14X_2 - 20X_3$

$Z = 250X_1 + 180X_2 + 300X_3 + 0X_4 + 0X_5$

$15X_3 = 2400 - 10X_1 - 12X_2 - X_4$

$$\boldsymbol{X_3 = 160 - \frac{2X_1}{3} - \frac{4X_2}{5} - \frac{X_4}{15}}$$

$$X_5 = 4800 - 24X_1 - 14X_2 - 20\left(160 - \frac{2X_1}{3} - \frac{4X_2}{5} - \frac{X_4}{15}\right) = 1600 - \frac{32X_1}{3} + 2X_2 - \frac{4X_4}{3}$$

$$Z = 250X_1 + 180X_2 + 300\left(160 - \frac{2X_1}{3} - \frac{4X_2}{5} - \frac{X_4}{15}\right) = 48000 + 50X_1 - 60X_2 - 20X_4$$

$X_3 = 160$, $X_5 = 4800$ $Z = 48000$

X_1 comes into the solution.

Best value = min{160/(2/3), 1600/(32/3)} = min {240, 150} = 150

X_1 replaces X_5.

$$\frac{32X_1}{3} = 1600 + 2X_2 + \frac{4X_4}{3} - X_5$$

$$X_1 = 150 + \frac{3X_2}{16} + \frac{X_4}{8} - \frac{3X_5}{32}$$

Optimum solution

$X_1 = 150$, $X_3 = 60$

Revenue = 55500

$$X_3 = 160 - \frac{2\left(150 + \frac{3X_2}{16} + \frac{X_4}{8} - \frac{3X_5}{32}\right)}{3} - \frac{4X_2}{5} - \frac{X_4}{15}$$

$$X_3 = 60 - \frac{37X_2}{40} - \frac{3X_4}{20} + \frac{X_5}{16}$$

$$Z = 48000 + 50\left(150 + \frac{3X_2}{16} + \frac{X_4}{8} - \frac{3X_5}{32}\right) - 60X_2 - 20X_4$$

$$Z = 55500 - \frac{405X_2}{8} - \frac{55}{4}X_4 - \frac{75X_5}{16}$$

Why did we not produce X_2 ?

We associate a worth for each resource. Let Y_1 and Y_2 be the worth of the two resources. Since the resources have been converted to products, the total revenue should be equal to the total worth of the resources.

When we produce, the unit revenue should be equal to unit worth
$10Y_1 + 24Y_2 = 250$, $15Y_1 + 20Y_2 = 300$; Solving, we get $Y_1 = 55/4$ $Y_2 = 75/16$
Total worth = 2400x55/4 + 4800x75/16 = 55500
For X_2, price = 180, value of resources = 12x55/4 + 14x75/16 = 1875/8 < 180 by 405/8

In general, we wish to find out Y_1 and Y_2 such that we minimize the total value of the resources and yet try to use them to meet the unit price. We therefore
Minimize $2400Y_1 + 4800Y_2$
such that $10Y_1 + 24Y_2 \geq 250$; $12Y_1 + 14Y_2 \geq 180$; $15Y_1 + 20Y_2 \geq 300$; Y_1 $Y_2 \geq 0$

The above problem is called the **dual** and has the solution $Y_1 = 55/4$ $Y_2 = 75/16$ with W = 55500

SIMPLEX SOLVES THE GIVEN PROBLEM (PRIMAL) AND ITS DUAL

Tabular method is used for quick hand computation and class room teaching

		$-X_1$	$-X_2$	$-X_3$
		-250	-180	-300
X_4	2400	10	12	15
X_5	4800	24	14	20

		$-X_1$	$-X_2$	$-X_4$
	48000	-50	60	20
X_3	160	2/3	4/5	1/15
X_5	1600	32/3	-2	-4/3

		$-X_5$	$-X_2$	$-X_4$
	55500	75/16	405/8	55/4
X_3	60	-1/16	37/40	3/20
X_1	150	3/32	-3/16	-1/8

Matrix method – How do I write a computer program for Simplex?

Maximize $250X_1 + 180X_2 + 300X_3$
$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$
$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_4 \\ X_5 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I; BX_B = b;\ X_B = B^{-1}b = b = \begin{bmatrix} 2400 \\ 4800 \end{bmatrix}$$

$$yB = C_{B;}y = C_B = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Find whether X_1 X_2 X_3 can enter the solution

$$C_1 - yP_1 = 250 - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 24 \end{bmatrix} = 250$$

$C_2 - yP_2 = 180$; $C_3 - yP_3 = 300$;

Variable X_3 enters. Takes value = min {2400/15, 4800/24} = 160.
Replaces X_4

Maximize $250X_1 + 180X_2 + 300X_3$

$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$

$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix} = B^{-1}b \qquad B = \begin{bmatrix} 15 & 0 \\ 20 & 1 \end{bmatrix}; = B^{-1} = \frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix}$$

$$X_B = \begin{bmatrix} X_3 \\ X_5 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix}\begin{bmatrix} 2400 \\ 4800 \end{bmatrix} = \begin{bmatrix} 160 \\ 1600 \end{bmatrix}$$

$$yB = C_{B;} y = C_B B^{-1} \qquad y = C_B B^{-1} = [300 \quad 0]\frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix} = [20 \quad 0]$$

Find whether X_1 X_2 X_4 can enter the solution

$$C_1 - yP_1 = 250 - [20 \quad 0]\begin{bmatrix} 10 \\ 24 \end{bmatrix} = 50$$

$C_2 - yP_2 = -60$; $C_4 - yP_3 = -20$;

Variable X_1 enters. New X_1 column is given by $B^{-1}P_1$.

$$B^{-1}P_1 = \frac{1}{15}\begin{bmatrix} 1 & 0 \\ -20 & 15 \end{bmatrix}\begin{bmatrix} 10 \\ 24 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 32/3 \end{bmatrix}$$

X_1 takes value = min {160/(2/3), 1600/(32/3) = 150 and replaces X_5

Maximize $250X_1 + 180X_2 + 300X_3$

$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$

$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = B^{-1}b \qquad B = \begin{bmatrix} 15 & 10 \\ 20 & 24 \end{bmatrix}; = B^{-1} = \frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix}$$

$$X_B = \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = \frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix}\begin{bmatrix} 2400 \\ 4800 \end{bmatrix} = \begin{bmatrix} 60 \\ 150 \end{bmatrix}$$

$$yB = C_{B;}y = C_B B^{-1} \qquad y = C_B B^{-1} = [300 \quad 250]\frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix} = \begin{bmatrix} \frac{55}{4} & \frac{75}{16} \end{bmatrix}$$

Find whether X_2 X_4 X_5 can enter the solution

$$C_2 - yP_2 = 180 - \begin{bmatrix} \frac{55}{4} & \frac{75}{16} \end{bmatrix}\begin{bmatrix} 12 \\ 14 \end{bmatrix} = \frac{-405}{8} \qquad C_4 - yP_4 = \frac{-55}{4}; \; C_5 - yP_5 = \frac{-75}{16}$$

No entering variable. Algorithm terminates

Variations in simplex algorithm – handling ≥ constraints

Minimize $4X_1 + 5X_2$
$X_1 + X_2 \geq 8$
$3X_1 + 5X_2 \geq 34$
$X_1, X_2 \geq 0$

$3X_1 + 5X_2 \geq 34$ becomes $3X_1 + 5X_2 - X_4 = 34$ with $X_4 \geq 0$. We cannot start simplex with X_4 as a basic variable because $X_4 = -34 + 3X_1 + 5X_2$ violates $X_4 \geq 0$

Two phase method – uses artificial variables and eliminates them in first phase

Big M method – uses artificial variables till the end

Dual simplex method – Starts with infeasible solution and reaches feasibility

Dual Simplex algorithm

Minimize $4X_1 + 5X_2$
$X_1 + X_2 \geq 8$
$3X_1 + 5X_2 \geq 34$
$X_1, X_2 \geq 0$

Minimize $4X_1 + 5X_2 + 0X_3 + 0X_4$
$X_1 + X_2 - X_3 \geq 8$
$3X_1 + 5X_2 - X_4 = 34$
$X_1, X_2, X_3, X_4 \geq 0$

$X_3 = -8 + X_1 + X_2$
$X_4 = -34 + 3X_1 + 5X_2$
$Z = 4X_1 + 5X_2$

$X_3 = -8$, $X_4 = -34$ $Z = 0$ is infeasible. To make it feasible one of the infeasible variables has to leave. We push out X_4 (most negative). X_2 comes in because the rate of increase of objective function is smaller – minimum {4/3, 5/5}

$X_3 = -8 + X_1 + X_2$
$X_4 = -34 + 3X_1 + 5X_2$
$Z = 4X_1 + 5X_2$

$5X_2 = 34 - 3X_1 - X_4$

$$\boldsymbol{X_2 = \frac{34}{5} - \frac{3X_1}{5} + \frac{X_4}{5}}$$

$$X_3 = -8 + X_1 + \left(\frac{34}{5} - \frac{3X_1}{5} + \frac{X_4}{5}\right) = \frac{-6}{5} + \frac{2X_1}{5} + \frac{X_4}{5}$$

$$Z = 40X_1 + 5\left(\frac{34}{5} - \frac{3X_1}{5} + \frac{X_4}{5}\right) = 34 + X_1 + X_4$$

$X_2 = 34/5$, $X_3 = -6/5$ $Z = 34$ is infeasible. To make it feasible, the infeasible variable has to leave. We push out X_3. X_1 comes in because the rate of increase of objective function is smaller – minimum {1/(2/5), 1/(1/5}.

$$X_3 = \frac{-6}{5} + \frac{2X_1}{5} + \frac{X_4}{5}$$

$$\frac{2}{5}X_1 = \frac{6}{5} + X_3 - \frac{X_4}{5}$$

$$X_1 = 3 + \frac{5}{2}X_3 - \frac{X_4}{2}$$

$$X_2 = \frac{34}{5} - \frac{3}{5}(3 + \frac{5}{2}X_3 - \frac{X_4}{2}) + \frac{X_4}{5}$$

$$X_2 = 5 - \frac{3}{2}X_3 + \frac{X_4}{2}$$

$$Z = 34 + X_1 + X_4 = 34 + 3 + \frac{5}{2}X_3 - \frac{X_4}{2} + X_4 = 37 + \frac{5}{2}X_3 + \frac{X_4}{2}$$

Limitations in simplex algorithm – cycling

Maximize $\frac{3}{4}X_1 - 20X_2 + \frac{1}{2}X_3 - 6X_4$

$$\frac{1}{4}X_1 - 8X_2 - X_3 + 9X_4 \le 0$$

$$\frac{1}{2}X_1 - 12X_2 - \frac{1}{2}X_3 + 3X_4 \le 0$$

$$X_3 \le 1$$

$$X_j \ge 0$$

Add X_5 X_6 X_7 as slack variables and start simplex.
Use Largest coefficient rule and first variable rule for entering and leaving variables

Successive basis are
$\{X_5, X_6, X_7\}$
$\{X_1, X_6, X_7\}$
$\{X_1, X_2, X_7\}$
$\{X_3, X_2, X_7\}$
$\{X_3, X_4, X_7\}$
$\{X_5, X_4, X_7\}$
$\{X_5, X_6, X_7\}$
Comes back to starting basis

Lexicographic rule
Smallest subscript rule (Bland)
$\{X_5, X_6, X_7\}$
$\{X_1, X_6, X_7\}$
$\{X_1, X_2, X_7\}$
$\{X_3, X_2, X_7\}$
$\{X_3, X_4, X_7\}$
$\{X_3, X_4, X_1\}$
$\{X_3, X_5, X_1\}$
$X_1 = X_3 = 1$ with $Z = 5/4$ is optimum

Making simplex algorithm faster and better

Memory and Speed

Memory – matrix method, column generation, bounded variables, decomposition

Speed – Time taken per iteration, number of iterations

Matrix inversion – Solving equations

- Substitution
- Gauss- Jordan
- Gaussian Elimination
- Eta factorization of the basis
- Sparse and dense matrices
- LU factorization

Eta factorization of the basis - example

Maximize $250X_1 + 180X_2 + 300X_3$
$10X_1 + 12X_2 + 15X_3 + X_4 = 2400$
$24X_1 + 14X_2 + 20X_3 + X_5 = 4800$

$$X_B = \begin{bmatrix} X_3 \\ X_1 \end{bmatrix} = B^{-1}b \qquad B_2 = \begin{bmatrix} 15 & 10 \\ 20 & 24 \end{bmatrix}; = B^{-1} = \frac{1}{160}\begin{bmatrix} 24 & -10 \\ -20 & 15 \end{bmatrix}$$

$$yB_2 = C_B; [y_1 \quad y_2]B_0E_1E_2 = [300 \quad 250] \qquad E_1 = \begin{bmatrix} 15 & 0 \\ 20 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 2/3 \\ 0 & 32/3 \end{bmatrix}$$

$$uE_2 = C_B; [u_1 \quad u_2]\begin{bmatrix} 1 & 2/3 \\ 0 & 32/3 \end{bmatrix} = [300 \quad 250]\, from\ which\ u = \begin{bmatrix} 300 & \frac{75}{16} \end{bmatrix}$$

$$yE_1 = u; [y_1 \quad y_2]\begin{bmatrix} 15 & 0 \\ 20 & 1 \end{bmatrix} = \begin{bmatrix} 300 & \frac{75}{16} \end{bmatrix} from\ which\ y = \begin{bmatrix} \frac{55}{4} & \frac{75}{16} \end{bmatrix}$$

Eta factorization of the basis - example

We do not explicitly invert the matrix but use the E matrices to substitute and get the values. We use this method to get y and entering column where we used $y = C_B B^{-1}$ and entering column = $B^{-1}P_j$

In the kth iteration $B_k = B_0 E_1 E_2 E_3 \ldots E_k$

Time per iteration of revised simplex = 32m + 10n
Time per iteration of standard simplex = mn/4
Revised simplex is better when $m > 100$

Average number of iterations of simplex (sample) – Avis and Chvatal (1978)

↓ m, n →	10	20	30	40	50
10	9.4	14.2	17.4	19.4	20.2
20		25.2	30.7	38	41.5
30			44.4	52.7	62.9
40				67.6	78.7
50					95.2

Largest coefficient rule

↓ m, n →	10	20	30	40	50
10	7.02	9.17	10.8	12.1	12.6
20		16.2	20.2	24.2	27.3
30			28.7	34.5	39.4
40				43.3	39.9
50					58.9

Largest increase rule

With $m \leq 50$ and $m + n \leq 200$, Dantzig (1963) reported that the number of iterations is usually $\leq 3m/2$, rarely going up to $3m$.

Complexity issues

1. Algebraic method – ${}^{n}C_{m}$ iterations.
2. Klee and Minty problems - 2^{n-1} iterations

$$Maximize \sum_{i=1}^{n} 10^{n-j} X_j$$

$$2\sum_{j=1}^{i-1} 10^{i-j} X_j + X_i \le 100^{i-1}$$

$X_j \ge 0.$

Maximize $100 X_1 + 10 X_2 + X_3$

Subject to $X_1 \le 1$

$20 X_1 + X_2 \le 100$

$200 X_1 + 20X_2 + X_3 \le 10000$

$X_1, X_2, X_3 \ge 0.$

Polynomially bounded algorithms

1. Ellipsoid Algorithm (1979)
2. Karmarkar's algorithm (1984)